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ORIGINAL PAPER

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Vibration and nonlinear dynamic response of eccentrically stiffened functionally graded composite truncated conical shells surrounded by an elastic medium in thermal environments

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Abstract A semi-analytical approach to eccentrically stiffened functionally graded truncated conical shells surrounded by an elastic medium in thermal environments is presented. Based on the classical thin shell theory with geometrical nonlinearity in von Karman Donnell sense, the smeared stiffeners technique and the Galerkin method, this paper deals with vibration and nonlinear dynamic problems. The truncated conical shells are reinforced by ring stiffeners made of full metal or full ceramic depending on the situation of the stiffeners at the metal-rich or ceramic-rich side of the shell, respectively. In addition, the study not only assume that the material properties depend on environment temperature variation, but also consider the thermal stresses in the stiffeners. Numerical results are given to evaluate effects of inhomogeneous, dimensional parameters, outside stiffeners, temperatures and elastic foundations on the vibration and nonlinear dynamic response of the structures.

Abbreviations

N	The volume fraction index (nonnegative number)
w	The deflection of the truncated conical shell
K_{w}	The Winkler foundation modulus
K_{p}	The shear layer foundation stiffness of the Pasternak model
$\varepsilon_S^0, \varepsilon_\theta^0$	The normal strains
$\gamma_{S\theta}^0$	The shear strain at the middle surface of the truncated conical shell
$k_S, k_\theta, k_{S\theta}$	The changes of curvatures and twist

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A_1, A_2	The cross-sectional area of eccentrically longitudinal and latitude stiffeners, respectively
d_1, d_2, h_1, h_2	The width and height of eccentrically longitudinal and latitude stiffeners, respectively
n_1, n_2	The numbers of eccentrically longitudinal and latitude stiffeners, respectively
E_0	Young's modulus of the stiffeners; $E_0 = E_c$ if the stiffeners are reinforced at the surface of the ceramic-rich, $E_0 = E_m$ if the stiffeners are reinforced at the surface of the metal-rich side

1 Introduction

As is well known, shells have increased structural stiffness compared to plates or panels. The advantage of shell structures is their capability in carrying loads and moments by a combined membrane and bending action due to their curvature. Advanced composite or functionally graded materials (FGMs) provide high performance and reliability due to their well-known characteristics. As a result, shell structures made of FGM will continue being widely used in various engineering fields such as aerospace, naval and industrial constructions, as well as sporting goods, medical devices, and many other areas. Moreover, FGM shells, like other composite structures, are usually reinforced by stiffening members to provide the benefit of added static and dynamic load-carrying capability with a relatively small additional weight penalty, or in other words, in order to provide material continuity and easy manufacturing, the FGM shells are reinforced by an eccentrically homogeneous stiffener system. At the moment, the investigation on static and dynamic of shell structures made of FGM has received extensive attention of many scientists. Shen [1] presented a post-buckling analysis for a functionally graded (FG) thin cylindrical shell of finite length subjected to compressive axial loads and in thermal environments. Darabi et al. [2] presented nonlinear analysis of dynamic stability for FG cylindrical shells under periodic axial loading based on large deflection theory, and Bolotin's method is then employed to obtain the steady-state vibrations for nonlinear Mathieu equations. Huang and Han [3] investigated nonlinear dynamic buckling of FG cylindrical shells subjected to time-dependent axial load. Sofiyev [4] presented an analytical study on the dynamic behavior of an infinitely-long, FGM cylindrical shell subjected to combined action of the axial tension, internal compressive load and ring-shaped compressive pressure with constant velocity. Zhang et al. [5] studied an analysis on the nonlinear dynamics of a clamped–clamped FGM circular cylindrical shell subjected to an external excitation and uniform temperature change. Duc [6] presented an analytical investigation on the nonlinear dynamic response of eccentrically stiffened FGM double-curved shallow shells resting on elastic foundations and subjected to axial compressive load and transverse load. Duc and Quan [7] investigated an analytical investigation on the nonlinear post-buckling for imperfect eccentrically stiffened FGM double-curved thin shallow shells on elastic foundation using a simple power-law distribution (P-FGM) in thermal environments. Duc and Thang [8] studied buckling of imperfect eccentrically stiffened metal-ceramic-metal S-FGM thin circular cylindrical shells with temperature-dependent properties in thermal environments. The same authors [9] presented the nonlinear response of imperfect eccentrically stiffened ceramic-metal-ceramic FGM circular cylindrical shells surrounded on elastic foundations and subjected to axial compression. Bich et al. [10] investigated a semi-analytical approach to investigate the nonlinear dynamic buckling of imperfect eccentrically stiffened FGM shallow shells taking into account the damping subjected to mechanical loads. Dung and Hoa [11,12] investigated nonlinear buckling and post-buckling behavior of FGM stiffened thin circular cylindrical shells subjected to external pressure and only under torsion load by the analytical approach. The effect of a Pasternak elastic foundation on the stability of EG orthotropic cylindrical shells including shear stresses subjected to a uniform hydrostatic pressure is investigated by Najafov et al. [13]. Tornabene et al. [14] compared 2D numerical approaches with an exact 3D shell solution in the case of free vibrations of FGM plates and shells in reference [14] and in [15] they solved numerically the free-vibration problem of sandwich shell structures with variable thickness and made of FGMs. Bani'c et al. [16] investigated the effect of the Winkler–Pasternak elastic foundation on the natural frequencies of carbon nanotube (CNT)-reinforced laminated composite plates and shells. The numerical analysis of laminated composite plates and shells resting on nonlinear elastic foundation is investigated by Tornabene et al. [17].

Truncated conical shells are known as one of the principal elements of structure in many technical fields. For instance, they are used for aircraft and satellites, submarines and water-borne ballistic missiles, or in civil engineering, they are frequently used too in containment vessels in elevated water tanks. In the open source literature, there are several authors studying the linear and nonlinear of conical cones and truncated cones structure made of different materials. Bhangale et al. [18] used finite element formulation based on first-order shear deformation theory to study the thermal buckling and vibration behavior of truncated FGM conical shells in a high-temperature environment. In this study, a Fourier series expansion for the displacement variable in the circumferential direction is used to model the FGM conical shell. Zhang and Li [19] discussed dynamic buckling of FGM truncated conical shells subjected to normal impact loads. In the analysis, the geometrically nonlinear large deformation and the initial imperfections are taken into account. The Galerkin procedure and Runge–Kutta integration scheme are used to solve nonlinear governing equations numerically. Zhao and Liew [20] presented free-vibration analysis of metal and ceramic functionally graded conical shell panels using the element-free kp-Ritz method. Sofiyev [21] investigated nonlinear vibration of truncated conical shells made of functionally graded materials (FGMs) using the large deformation theory with von Karman–Donnell type of kinematic nonlinearity. By using the superposition method, Galerkin method and Harmonic balance method, the nonlinear vibration of an FGM truncated conical shell is analyzed. Setoodeh et al. [22] focused on the transient dynamic and free-vibration analysis of FG axisymmetric truncated conical shells with non-uniform thickness. Two numerically efficient and accurate solution methods are presented to study the transient dynamic responses of FG shells subjected to either internal or external mechanical shock loading. Najafov and Sofiyev [23] investigated nonlinear dynamic analysis of FG-truncated conical shells surrounded by an elastic medium using the large deformation theory with von Karman–Donnell type of kinematic nonlinearity. The Pasternak model is used to describe the reaction of the elastic foundation on the FG conical shell. Sofiyev and Kuruoglu [24] studied nonlinear buckling of a truncated conical shell made of FGMs surrounded by an elastic medium using the large deformation theory with von Karman–Donnell type of kinematic nonlinearity. A two-parameter foundation model (Pasternak-type) is used to describe the shell–foundation interaction. Sofiyev [25] investigated the vibration and stability of FG conical shells under a compressive axial load using the shear deformation theory. Sofiyev and Kuruoglu [26] obtained a closed form of the solution for critical combined loads of an FG-truncated conical shell in the framework of the shear deformation theory. Yang et al. [27] investigated nonlinear dynamic behaviors of ceramic–metal graded truncated conical shell subjected to complex loads. Jabbari et al. [28] presented thermoelastic analysis of axially functionally graded rotating thick truncated conical shells with varying thickness. Jooybar et al. [29] investigated influences of thermal environment on the free-vibration characteristics of FG-truncated conical panels based on the first-order shear deformation theory. Castro et al. [30] studied linear buckling predictions of unstiffened laminated composite cylinders and cones under various loading and boundary conditions using semi-analytical models. Asemi et al. [31] considered a thick truncated hollow cone with finite length made of two-dimensional functionally graded materials (2D-FGMs) subjected to combined loads as internal, external and axial pressure. Jam and Kiani [32] presented linear buckling analysis for nano-composite conical shells reinforced with single-walled carbon nanotubes subjected to lateral pressure. Tornabene [33] studied the dynamic behavior of moderately thick functionally graded conical, cylindrical shells and annular plates based on the first-order shear deformation theory. Dung et al. [34] investigated mechanical buckling load of ES-FGM truncated conical shells subjected to axial compressive load and external uniform pressure; the stability equations of the conical shell were derived using the adjacent equilibrium criterion. Duc et al. [35] investigated the linear stability analysis of ES-FGM conical shell panels reinforced by mechanical and thermal loads on elastic foundations. Duc and Cong [36] studied the stability of an ES-FGM truncated conical shell surrounded on elastic foundations under thermal loads with both FGM shell and stiffeners having temperature-dependent properties and by using the first-order shear deformation theory. Dung and Chan [37] investigated the mechanical buckling of FGM thick truncated conical shells reinforced by stringers and rings and subjected to axial compressive load and uniform external pressure load. Duc et al. [38] presented an analytical approach to investigate the mechanical and thermal buckling of FGMs sandwich truncated conical shells resting on Pasternak elastic foundations, subjected to thermal load and axial compressive load.

To the best of the authors' knowledge, research on nonlinear dynamic analysis of ES-FGM truncated conical shells has not yet been addressed. Therefore, this paper aims to investigate the nonlinear dynamic response of ES-FGM truncated conical shells in thermal environments by a stress function. The shells are reinforced by eccentrically longitudinal stiffeners made of full metal or full ceramic depending on the location of the stiffeners at the metal-rich or ceramic-rich side of the shell, respectively. In addition, the study not only assumes that the material properties depend on environment temperature variation, but also considers the thermal deformation of the stiffeners. The Result and Discussion section analyzes and discusses the effects of material an geometrical properties, elastic foundations and eccentric stiffeners on the dynamic response of ES-FGM truncated conical shells.

1.1 Theoretical formulations

Consider an ES-FGM truncated conical shell resting on elastic foundations; the shell is of thickness h, and radii $R_1 < R_2$, length L and the semi-vertex angle of the cone γ. The meridional, circumferential and normal

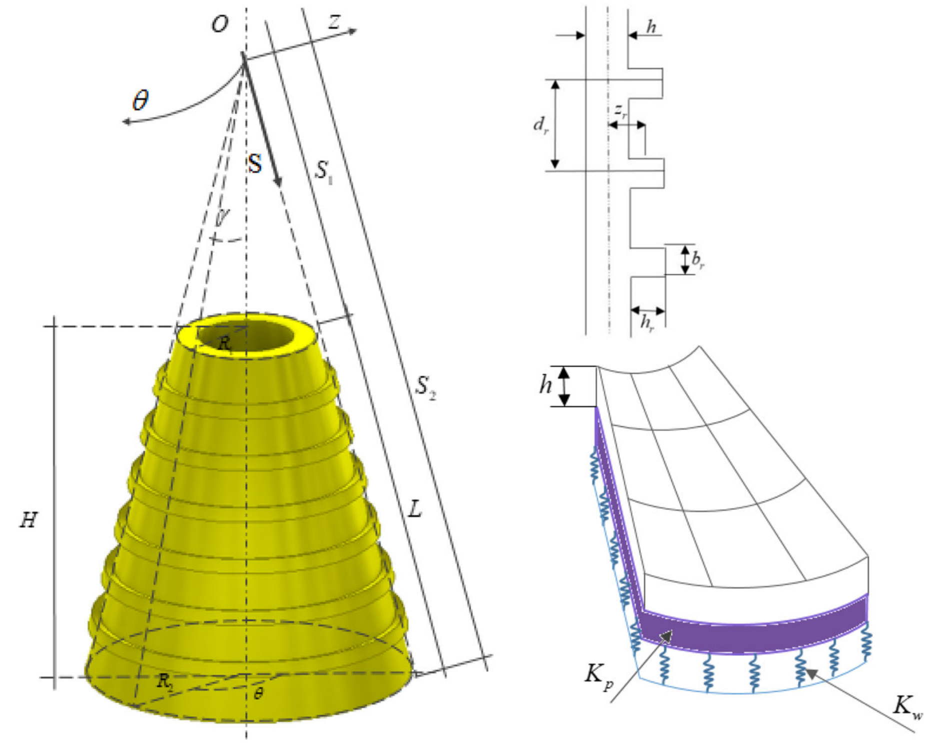

Fig. 1 Configuration of an ES-FGM truncated conical shell

directions of the shell are denoted by S, θ and z, respectively. A schematic of the shell with the assigned coordinate system and geometric characteristics are shown in Fig. 1. S_1, S_2 are the distances from the vertex to the small and large bases, respectively. Also, u, v and w denote displacement (due to loads) of a point in the middle surface in the direction of a generator, the circumferential direction and the inward normal direction, respectively.

1.2 The Winkler–Pasternak elastic foundations

The ES-FGM truncated conical shell rests on the elastic foundation. For the elastic foundation, one assumes the two-parameter elastic foundation model proposed by Pasternak. The foundation medium is assumed to be linear, homogeneous and isotropic. The bonding between the truncated conical shell and the foundation is perfect and frictionless. If the effects of damping and inertia force in the foundation are neglected, the shell–foundation interaction can be represented as

$$q_e = K_\mathrm{w} w - K_\mathrm{p}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right),$$

where q_e is the force per unit area, K_w (N/m^3) is the spring parameter of the one-parameter elastic foundation (or Winkler foundation parameter) and K_p(N/m) is the shear parameter of the two-parameter elastic foundations (or Pasternak-type elastic foundation), w is the displacement of the middle surface in the normal direction, positive toward the axis of the cone and assumed to be much smaller than the thickness. Note that by setting $K_p = 0$, the Pasternak model becomes that of the Winkler foundation model.

1.3 The material properties of functionally graded shells

It is well-known that FGMs are microscopically inhomogeneous materials, in which material properties vary smoothly and continuously from one surface to the other. In the present paper, the FGM shell is assumed to be made from a mixture of two component material which are metal and ceramic and the volume fraction of the constituent materials is supposed to vary constantly as a power-law form according to the thickness direction:

$$V_c(z) = \left(\frac{2z+h}{2h}\right)^N, \quad -\frac{h}{2} \le z \le \frac{h}{2},$$
$$V_m(z) = 1 - V_c(z). \tag{1}$$

Here, subscripts m and c refer to the metal and ceramic constituents, respectively, and nonnegative number $N \ge 0$ is volume fraction index that defines the distribution of constituents in FGM.

The material properties of the ES-FGM truncated conical shell such as the elasticity modulus E, the Poisson ratio ν and the coefficient of thermal expansion α of FGM can be rewritten from the formulation (1) as below:

$$\begin{bmatrix} E(z,T) \\ \nu(z,T) \\ \alpha(z,T) \end{bmatrix} = \begin{bmatrix} E(T) \\ \nu_m(T) \\ \alpha_m(T) \end{bmatrix} + \begin{bmatrix} E_{cm}(T) \\ \nu_{cm}(T) \\ \alpha_{cm}(T) \end{bmatrix} \left(\frac{2z+h}{h}\right)^N \qquad N \ge 0, \quad -\frac{h}{2} \le z \le \frac{h}{2}, \tag{2}$$

where $E_{cm}(T) = E_c(T) - E_m(T)$, $\nu_{cm}(T) = \nu_c(T) - \nu_m(T)$, $\alpha_{cm}(T) = \alpha_c(T) - \alpha_m(T)$. Furthermore, specialization of Eq. (2) for $N = 0$ gives corresponding properties of isotropic ceramic shells, and the percentage of the metal constituent in the FGM shell is enhanced as N index increases. As N goes to infinity, Eq. (2) turns to corresponding properties of pure metal truncated conical shell.

In order to analyze the influence of temperature on the eccentrically stiffened FGM truncated conical shell, the study is not only assumed that the material properties depend on environment temperature variation, but also considered the thermal deformation of stiffeners. Therefore, the stiffener's geometry parameters after undergoing thermal deformation process can be defined as

$$h_r^T = h_r(1+\alpha_r\Delta T), \quad z_r^T = z_r(1+\alpha_r\Delta T), \quad b_r^T = b_r(1+\alpha_r\Delta T), \quad d_r^T = d_r(1+\alpha_r\Delta T), \tag{3}$$

where h_r^T, z_r^T, b_r^T, d_r^T are the new geometrical dimension of stiffeners due to heat expansion, ΔT is the environmental temperature variation, and α_r is the thermal expansion coefficient of the material composed stiffeners.

2 Constitutive relations and governing equations

According to the classical shell theory, the strains at the middle surface and the change of curvatures and twist are related to the displacement components u, v, w in the S, θ, z coordinate directions, respectively, taking into account Von Karman–Donnell nonlinear terms as [9,40]

$$\begin{aligned}
\varepsilon_S^0 &= \tfrac{\partial u}{\partial S} + \tfrac{1}{2}\left(\tfrac{\partial w}{\partial S}\right)^2, & k_S &= -\tfrac{\partial^2 w}{\partial S^2}, \\
\varepsilon_\theta^0 &= \tfrac{1}{S}\tfrac{\partial v}{\partial \phi} + \tfrac{u}{S} - \tfrac{w}{S}\cos(\gamma) + \tfrac{1}{2S^2}\left(\tfrac{\partial w}{\partial \phi}\right)^2, & k_\theta &= -\tfrac{1}{S^2}\tfrac{\partial^2 w}{\partial \phi^2} - \tfrac{1}{S}\tfrac{\partial w}{\partial S}, \\
\gamma_{S\theta}^0 &= 2\left[\tfrac{1}{S}\tfrac{\partial u}{\partial \phi} - \tfrac{v}{S} + \tfrac{\partial v}{\partial S} + \tfrac{1}{S}\left(\tfrac{\partial w}{\partial S}\tfrac{\partial w}{\partial \phi}\right)^2\right], & k_{S\theta} &= -\tfrac{1}{S}\tfrac{\partial^2 w}{\partial S \partial \phi} + \tfrac{1}{S^2}\tfrac{\partial w}{\partial \phi}.
\end{aligned} \tag{4}$$

where $\phi = \theta \sin(\gamma)$.

The strains across the shell thickness at a distance z from the mid-plane are:

$$\varepsilon_S = \varepsilon_S^0 + zk_S, \quad \varepsilon_\theta = \varepsilon_\theta^0 + zk_\theta, \quad \gamma_{S\theta} = \gamma_{S\theta}^0 + 2zk_{S\theta}. \tag{5}$$

For stiffeners in thermal environment, the stress may be inferred from [11,12]:

$$\sigma_S^s = E_s\varepsilon_S - \frac{E_s}{1-2\nu_s}\alpha_s\Delta T, \quad \sigma_\theta^r = E_r\varepsilon_\theta - \frac{E_s}{1-2\nu_r}\alpha_r\Delta T. \tag{6}$$

Using Eqs. (4) and (5), the geometrical compatibility equation for a truncated conical shell is indicated as [9]

$$\begin{aligned}
&\tfrac{\cos(\gamma)}{S}\tfrac{\partial^2 w}{\partial S^2} - \tfrac{1}{S}\tfrac{\partial^2 \gamma_{S\theta}^0}{\partial S \partial \phi} - \tfrac{1}{S^2}\tfrac{\partial \gamma_{S\theta}^0}{\partial \phi} + \tfrac{\partial^2 \varepsilon_\theta^0}{\partial S^2} + \tfrac{1}{S^2}\tfrac{\partial^2 \varepsilon_S^0}{\partial \phi^2} + \tfrac{2}{S}\tfrac{\partial \varepsilon_\theta^0}{\partial S} - \tfrac{1}{S}\tfrac{\partial \varepsilon_S^0}{\partial S} \\
&= \tfrac{1}{S^4}\left(\tfrac{\partial w}{\partial \phi}\right)^2 - \tfrac{2}{S^3}\tfrac{\partial w}{\partial \phi}\tfrac{\partial^2 w}{\partial S \partial \phi} - \tfrac{1}{S^2}\left[\tfrac{\partial^2 w}{\partial S^2}\tfrac{\partial^2 w}{\partial \phi^2} - \left(\tfrac{\partial^2 w}{\partial S \partial \phi}\right)^2\right] - \tfrac{1}{S}\tfrac{\partial w}{\partial S}\tfrac{\partial^2 w}{\partial S^2},
\end{aligned} \tag{7}$$

The stress–strain relationships for the truncated conical shell including temperature effect are defined by Hooke's law as

$$\sigma_S = \frac{E}{1-\nu^2}\left[\varepsilon_S^0 + zk_S + \nu\left(\varepsilon_\theta^0 + zk_\theta\right) - (1+\nu)\,\alpha\Delta T\right],$$
$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\varepsilon_\theta^0 + zk_\theta + \nu\left(\varepsilon_S^0 + zk_S\right) - (1+\nu)\,\alpha\Delta T\right], \quad \sigma_{S\theta} = \frac{E}{2(1+\nu)}\left(\gamma_{S\theta}^0 + 2zk_{S\theta}\right),$$
$$\sigma_S^s = E_s\left(\varepsilon_S^0 + zk_S\right) - \frac{E_s}{1-2\nu_s}\alpha_s\Delta T, \quad \sigma_\theta^r = E_r\left(\varepsilon_\theta^0 + zk_\theta\right) - \frac{E_s}{1-2\nu_r}\alpha_r\Delta T. \tag{8}$$

The force and moment resultants of an ES-FGM truncated conical shell are expressed in terms of the stress components through the thickness as

$$N_S = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_S \mathrm{d}z, \qquad M_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta z\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_r} \sigma_\theta^{st}\frac{b_r^T}{d_r^T} z\mathrm{d}z,$$
$$N_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta \mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_r} \sigma_\theta^{st}\frac{b_r^T}{d_r^T}\mathrm{d}z, \qquad M_{S\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{S\theta} z\mathrm{d}z, \tag{9}$$
$$N_{S\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{S\theta}\mathrm{d}z, \qquad M_S = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_S z\mathrm{d}z.$$

By setting Eq. (4) into Eq. (9), Eq. (9) can be rewritten in detail as

$$N_S = A_{11}\varepsilon_S^0 + A_{12}\varepsilon_\theta^0 + B_{11}k_S + B_{12}k_\theta + \Phi_1;\ N_\theta = A_{21}\varepsilon_S^0 + A_{22}\varepsilon_\theta^0 + B_{21}k_S + B_{22}k_\theta + \Phi_2;$$
$$N_{S\theta} = A_{33}\gamma_{S\theta}^0 + B_{33}k_{S\theta};\ M_S = C_{11}\varepsilon_S^0 + C_{12}\varepsilon_\theta^0 + D_{11}k_S + D_{12}k_\theta + \Phi_3;$$
$$M_\theta = C_{21}\varepsilon_S^0 + C_{22}\varepsilon_\theta^0 + D_{21}k_S + D_{22}k_\theta + \Phi_4; \quad M_{S\theta} = C_{33}\gamma_{S\theta}^0 + D_{33}k_{S\theta}. \tag{10}$$

From the three first equation of (10),

$$\varepsilon_\theta^0 = A_{11}^* N_\theta + A_{12}^* N_S + B_{11}^* k_S + B_{12}^* k_\theta + \Psi_1, \quad \varepsilon_S^0 = A_{21}^* N_\theta + A_{22}^* N_S + B_{21}^* k_S + B_{22}^* k_\theta + \Psi_2,$$
$$\gamma_{S\theta}^0 = A_{33}^* N_{S\theta} + B_{33}^* k_{S\theta}, \tag{11}$$

in which

$$A_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2}\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_s} E_s\frac{b_s^T}{d_s^T}\mathrm{d}z;\ A_{12} = A_{21} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\nu}{1-\nu^2}\mathrm{d}z;\ A_{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2}\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_r} E_r\frac{b_r^T}{d_r^T}\mathrm{d}z;$$
$$A_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{2(1+\nu)}\mathrm{d}z;\ B_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2}z\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_s} E_s\frac{b_s^T}{d_s^T}z\mathrm{d}z;\ B_{12} = B_{21} = C_{12} = C_{21} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\nu}{1-\nu^2}z\mathrm{d}z;$$
$$B_{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2}z\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_r} E_r\frac{b_r^T}{d_r^T}z\mathrm{d}z;\ B_{33} = C_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{(1+\nu)}z\mathrm{d}z;\ C_{11} = B_{11};\ C_{22} = B_{22};$$
$$D_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2}z^2\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_s} E_s\frac{b_s^T}{d_s^T}z^2\mathrm{d}z;\ D_{12} = D_{21} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\nu}{1-\nu^2}z^2\mathrm{d}z;\ D_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{(1+\nu)}z^2\mathrm{d}z;$$
$$D_{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{1-\nu^2}z^2\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_r} E_r\frac{b_r^T}{d_r^T}z^2\mathrm{d}z;\ \Phi_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{-E(1+\nu)}{1-\nu^2}\alpha\Delta T\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_s}$$
$$\frac{-E_s}{1-2\nu_s}\frac{b_s^T}{d_s^T}\alpha_s\Delta T\mathrm{d}z;$$
$$\Phi_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{-E(1+\nu)}{1-\nu^2}\alpha\Delta T\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_r} \frac{-E_r}{1-2\nu_r}\frac{b_r^T}{d_r^T}\alpha_r\Delta T\mathrm{d}z;\ \Phi_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{-E(1+\nu)}{1-\nu^2}\alpha\Delta Tz\mathrm{d}z + \int_{-\frac{h}{2}}^{\frac{h}{2}+h_s}$$
$$\frac{-E_s}{1-2\nu_s}\frac{b_s^T}{d_s^T}\alpha_s\Delta Tz\mathrm{d}z;$$

$$\Phi_4 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{-E\,(1+\nu)}{1-\nu^2}\alpha\Delta T z\mathrm{d}z + \int_{\frac{h}{2}}^{\frac{h}{2}+h_r} \frac{-E_r}{1-2\nu_r}\frac{b_r^T}{d_r^T}\alpha_r\Delta T z\mathrm{d}z,$$
$$\Delta = A_{22}A_{11} - A_{12}A_{21},\ A_{33}^* = \frac{1}{A_{33}},\ B_{33}^* = -\frac{B_{33}}{A_{33}},\ A_{21}^* = \frac{-A_{12}}{\Delta},\ A_{22}^* = \frac{A_{22}}{\Delta},\ A_{11}^* = \frac{A_{11}}{\Delta},\ A_{12}^* = \frac{-A_{21}}{\Delta},$$
$$B_{21}^* = \frac{A_{12}B_{21} - B_{11}A_{22}}{\Delta},\ B_{22}^* = \frac{A_{12}B_{22} - B_{12}A_{22}}{\Delta},\ B_{11}^* = \frac{B_{11}A_{21} - A_{11}B_{21}}{\Delta},\ B_{12}^* = \frac{B_{12}A_{21} - A_{11}B_{22}}{\Delta},$$
$$\Psi_1 = \frac{\Phi_1A_{21} - A_{11}\Phi_2}{\Delta},\ \Psi_2 = \frac{A_{12}\Phi_2 - \Phi_1A_{22}}{\Delta}.$$

By substitution of results in (11) into the equation of (10), we receive

$$M_S = C_{11}^*N_\theta + C_{12}^*N_S + D_{11}^*k_S + D_{12}^*k_\theta + X_1,\quad M_\theta = C_{21}^*N_\theta + C_{22}^*N_S + D_{21}^*k_S + D_{22}^*k_\theta + X_2,$$
$$M_{S\theta} = C_{33}^*N_{S\theta} + D_{33}^*k_{S\theta}. \tag{12}$$

Here,

$$C_{11}^* = \left(C_{11}A_{21}^* + C_{12}A_{11}^*\right),\quad C_{12}^* = \left(C_{11}A_{22}^* + C_{12}A_{12}^*\right),\quad D_{11}^* = \left(C_{11}B_{21}^* + C_{12}B_{11}^* + D_{11}\right),$$
$$D_{12}^* = \left(C_{11}B_{22}^* + C_{12}B_{12}^* + D_{12}\right),\quad C_{21}^* = \left(C_{21}A_{21}^* + C_{22}A_{11}^*\right),\quad C_{22}^* = \left(C_{21}A_{22}^* + C_{22}A_{12}^*\right),$$
$$D_{21}^* = \left(C_{21}B_{21}^* + C_{22}B_{11}^* + D_{21}\right),\quad D_{22}^* = \left(C_{21}B_{22}^* + C_{22}B_{12}^* + D_{22}\right),\quad C_{33}^* = C_{33}A_{33}^*,$$
$$D_{33}^* = \left(D_{33} + C_{33}B_{33}^*\right),\quad X_1 = C_{11}\Psi_2 + C_{12}\Psi_1 + \Phi_3,\quad X_2 = C_{21}\Psi_2 + C_{22}\Psi_1 + \Phi_4.$$

The nonlinear motion equations of the truncated conical shell which is subjected to the external force and surrounded by the elastic foundation based on the classical shell theory [23] is

$$S\frac{\partial N_S}{\partial S} + \frac{\partial N_{S\theta}}{\partial \phi} + N_S - N_\theta = I_0\frac{\partial^2 u}{\partial t^2},$$
$$\frac{\partial N_\theta}{\partial \phi} + S\frac{\partial N_{S\theta}}{\partial S} + 2N_{S\theta} = I_0\frac{\partial^2 v}{\partial t^2}, \tag{13}$$
$$S\frac{\partial^2 M_S}{\partial S^2} - \frac{\partial M_\theta}{\partial S} + 2\left(\frac{\partial^2 M_{S\theta}}{\partial S\partial\theta} + \frac{1}{S}\frac{\partial M_{S\theta}}{\partial\theta}\right) + \frac{1}{S}\frac{\partial^2 M_\theta}{\partial\phi^2} - N_\theta\cot\gamma + \frac{\partial}{\partial S}\left(SN_S\frac{\partial w}{\partial S} + N_{S\theta}\frac{\partial w}{\partial\phi}\right) +$$
$$+2\frac{\partial M_S}{\partial S} + q - SK_w w + \frac{\partial}{\partial\phi}\left(N_{S\theta}\frac{\partial w}{\partial S} + \frac{N_\theta}{S}\frac{\partial w}{\partial\phi}\right) + SK_p\left(\frac{\partial^2 w}{\partial S^2} + \frac{1}{S}\frac{\partial w}{\partial S} + \frac{1}{S^2}\frac{\partial^2 w}{\partial\phi^2}\right) = I_0\frac{\partial^2 w}{\partial t^2},$$

with $I_0 = \int_{-h/2}^{h/2}\rho\mathrm{d}z$, q is an external force uniformly distributed on the surface of the shell.

By using Volmir's assumption [39], the displacements u and v are extremely small compared to the deflection w deflection, it leads to the inertia forces $I_0\frac{\partial^2 u}{\partial t^2} \to 0$, $I_0\frac{\partial^2 v}{\partial t^2} \to 0$ and can be ignored.

Equations (12) and (13) will be satisfied identically by using the stress function

$$N_S = \frac{1}{S^2}\frac{\partial^2 F}{\partial\phi^2} + \frac{1}{S}\frac{\partial F}{\partial S},\quad N_\theta = \frac{\partial^2 F}{\partial S^2},\quad N_{S\theta} = -\frac{1}{S}\frac{\partial^2 F}{\partial S\partial\phi} + \frac{1}{S^2}\frac{\partial F}{\partial\phi}. \tag{14}$$

Substituting Eq. (5) into Eq. (12), then taking the result with the formulation (14) into the third equation of motion (13) yields

$$\frac{C_{22}^*}{S^2}\frac{\partial F}{\partial S} + \frac{-C_{22}^*S^2 - \cot\gamma S^3}{S^3}\frac{\partial^2 F}{\partial S^2} + \left(2C_{11}^* - C_{21}^* + C_{12}^*\right)\frac{\partial^3 F}{\partial S^3} + SC_{11}^*\frac{\partial^4 F}{\partial S^4} + \frac{\left(2C_{12}^* + 2C_{22}^* - 2C_{33}^*\right)}{S^3}\frac{\partial^2 F}{\partial\phi^2}$$
$$+ \frac{C_{22}^*}{S^3}\frac{\partial^4 F}{\partial\phi^4} + \frac{2C_{33}^* - 2C_{12}^*}{S^2}\frac{\partial^3 F}{\partial S\partial\phi^2} + \frac{C_{12}^* - 2C_{33}^* + C_{21}^*}{S}\frac{\partial^4 F}{\partial S^2\partial\phi^2} + \left(-SK_w\right)w + \left(K_p + \frac{-D_{22}^*}{S^2}\right)\frac{\partial w}{\partial S}$$
$$+ \left(SK_p + \frac{1}{S}D_{22}^*\right)\frac{\partial^2 w}{\partial S^2} + \left(D_{21}^* - 2D_{11}^* - D_{12}^*\right)\frac{\partial^3 w}{\partial S^3} + \left(-SD_{11}^*\right)\frac{\partial^4 w}{\partial S^4} + \frac{K_pS^2 - 2D_{33}^* - 2D_{12}^* - 2D_{22}^*}{S^3}\frac{\partial^2 w}{\partial\phi^2} +$$
$$-\frac{D_{22}^*}{S^3}\frac{\partial^4 w}{\partial\phi^4} + \frac{\left(2D_{12}^* + 2D_{33}^*\right)}{S^2}\frac{\partial^3 w}{\partial S\partial\phi^2} - \frac{D_{12}^* + 2D_{33}^* + D_{21}^*}{S}\frac{\partial^4 w}{\partial S^2\partial\phi^2} + \left(\frac{2}{S^2}\frac{\partial^2 F}{\partial S\partial\phi} - \frac{2}{S^3}\frac{\partial F}{\partial\phi}\right)\frac{\partial w}{\partial\phi} +$$

$$+\frac{\partial^2 F}{\partial S^2}\frac{\partial w}{\partial S}+\left(\frac{1}{S}\frac{\partial^2 F}{\partial \phi^2}+\frac{\partial F}{\partial S}\right)\frac{\partial^2 w}{\partial S^2}+\frac{1}{S}\frac{\partial^2 F}{\partial S^2}\frac{\partial^2 w}{\partial \phi^2}+\left(\frac{2}{S^2}\frac{\partial F}{\partial \phi}-\frac{2}{S}\frac{\partial^2 F}{\partial S\partial \phi}\right)\frac{\partial^2 w}{\partial S\partial \phi}+q=I_0\frac{\partial^2 w}{\partial t^2}. \tag{15}$$

To simplify the calculation process, a transformation is intended to take the forms (7), (15) into differential equations with constant coefficient. The transformation is supposed such as [18]

$$S=S_1e^x,\quad F=F_1e^{2x}. \tag{16}$$

Setting formulations (16) into Eq. (15) and establishing a lot of calculations lead to the transformed Eqs. (17). To obtain Eq. (18), first substituting expression (11) into formulation (7) and then using the function (14), and finally applying the formulations (16) leads to

$$L_{11}(F_1)+L_{12}(\mathrm{w})+L_{13}(F_1,w)+q=I_0\frac{\partial^2 w}{\partial t^2}, \tag{17}$$

$$L_{21}(F_1)+L_{22}(w)+L_{23}(w,w)=0, \tag{18}$$

where L_{ij} are given in Appendix A.

The equation system (17) and (18) is the fundamental system used to solve the nonlinear dynamic problems of the ES-FGM truncated conical shell. It is an equation system dependent on the unknown variables w and F.

3 Solutions

Supposing that the ES-FGM truncated conical shell is simply supported along the periphery of the edge subjected to uniformly distributed load in the temperature environment. The solution for Eq. (18) could be approximately assumed as form [4,21]

$$w=fe^x\left[\sin(\beta_1 x)\sin(\beta_2\alpha)+G\sin^2(\beta_1 x)\right],\quad \beta_1=\frac{m\pi}{x_0},\quad \beta_2=\frac{n}{\sin(\gamma)}. \tag{19}$$

Here f and G are the unknowns which denote the linear and nonlinear part of the deflection function w, respectively. m is the number of half waves along the generatrix and n is the number of full-waves in circumferential direction. The form of this approximate solution which satisfies the geometric boundary condition of $w=0$ at $x=0$ and $x=x_0$ was proposed by Sofiyev in [21] for FGM truncated conical shells.

By setting the approximate solution (19) with its boundary condition into Eq. (18) and appling the superposition method as [21], we receive the solution yield as

$$\begin{aligned}F_1=&f(t)K_1\mathrm{e}^{-x}\sin(B_1x)\sin(B_2\varphi)+f(t)K_2\mathrm{e}^{-x}\cos(B_1x)\sin(B_2\varphi)+f(t)K_3G\mathrm{e}^{-x}\cos(2B_1x)\\&+f(t)K_4G\mathrm{e}^{-x}\sin(2B_1x)+\left(f^2(t)G^2K_{51}+f^2(t)K_{52}+f(t)GK_{53}\right)\cos(2B_1x)+\\&+\left(f^2(t)G^2K_{61}+f^2(t)K_{62}+f(t)GK_{63}\right)\sin(2B_1x)+f^2(t)K_7\cos(2B_1x)\cos(2B_2\varphi)\\&+f^2(t)K_8\sin(2B_1x)\cos(2B_2\varphi)+\left(f^2(t)GK_{91}+f(t)K_{92}\right)\cos(B_1x)\sin(B_2\varphi)+\\&+\left(f^2(t)GK_{101}+f(t)K_{102}\right)\sin(B_1x)\sin(B_2\varphi)+f^2(t)GK_{11}\cos(3B_1x)\sin(B_2\varphi)+\\&+f^2(t)GK_{12}\sin(3B_1x)\sin(B_2\varphi)+f^2(t)G^2K_{13}\cos(4B_1x)+f^2(t)G^2K_{14}\sin(4B_1x)+\\&+f^2(t)K_{15}\cos(2B_2\varphi)+f(t)GK_{16}\mathrm{e}^{-x}-(1/2)\left(1+\mathrm{e}^{2x}+\cos(B_2\varphi)\right)T_{\mathrm{force}}S_1^2,\end{aligned} \tag{20}$$

where T_{force} is the axial force and the coefficients $K_i\,(i=1,\ldots,16)$ depend on the material properties and the original geometrical shape of the shell a listed in detail below, where a_{ij} may be found in Appendix B:

$$K_1=\frac{-(a_{14}a_{16}+a_{15}a_{17})}{a_{14}^2+a_{15}^2};\ K_2=\frac{(a_{14}a_{17}-a_{15}a_{16})}{a_{14}^2+a_{15}^2};\ K_3=\frac{-(a_{26}a_{28}+a_{27}a_{29})}{a_{26}^2+a_{27}^2};\ K_4=\frac{(a_{26}a_{29}-a_{27}a_{28})}{a_{26}^2+a_{27}^2};$$

$$K_{51}=\frac{-(a_{30}a_{32}+a_{31}a_{33})}{a_{30}^2+a_{31}^2};\ K_{52}=\frac{-(a_{34}a_{36}+a_{35}a_{37})}{a_{34}^2+a_{35}^2};\ K_{53}=\frac{-(a_{38}a_{40}+a_{39}a_{41})}{a_{38}^2+a_{39}^2};\ K_{61}=\frac{(a_{30}a_{33}-a_{31}a_{32})}{a_{30}^2+a_{31}^2};$$

$$K_{62}=\frac{(a_{34}a_{37}-a_{35}a_{36})}{a_{34}^2+a_{35}^2};\ K_{63}=\frac{(a_{38}a_{41}-a_{39}a_{40})}{a_{38}^2+a_{39}^2};\ K_7=\frac{-(a_{10}a_{12}+a_{11}a_{13})}{a_{10}^2+a_{11}^2};\ K_8=\frac{(a_{10}a_{13}-a_{11}a_{12})}{a_{10}^2+a_{11}^2};$$

$$K_{91}=\frac{(a_{18}a_{21}-a_{19}a_{20})}{a_{18}^2+a_{19}^2};\ K_{92}=\frac{(a_{22}a_{25}-a_{23}a_{24})}{a_{22}^2+a_{23}^2};\ K_{101}=\frac{-(a_{18}a_{20}+a_{19}a_{21})}{a_{18}^2+a_{19}^2};$$
$$K_{102}=\frac{-(a_{22}a_{24}+a_{23}a_{25})}{a_{22}^2+a_{23}^2};\quad K_{11}=\frac{-(a_6a_8+a_7a_9)}{a_6^2+a_7^2},\ K_{12}=\frac{(a_6a_9-a_7a_8)}{a_6^2+a_7^2},\ K_{13}=\frac{-(a_1a_3+a_2a_4)}{a_1^2+a_2^2}$$
$$K_{14}=\frac{(a_1a_4-a_2a_3)}{a_1^2+a_2^2},\ K_{15}=a_5,\ K_{16}=\frac{-a_{43}}{a_{42}}.$$

The Galerkin method is applied twice to solve the nonlinear dynamic problem (17). In particular, the first time:

$$\begin{aligned}&\int_0^{x_0}\int_0^{2\pi\sin(\gamma)}\left(L_{11}(F_1)+L_{12}(\mathrm{w})+L_{13}(F_1,w)+q\right)\mathrm{e}^x\sin(B_1x)\sin(B_2\varphi)\,d\varphi dx\\&=\int_0^{x_0}\int_0^{2\pi\sin(\gamma)} I_0\frac{\partial^2 w}{\partial t^2}\mathrm{e}^x\sin(B_1x)\sin(B_2\varphi)\,d\varphi dx,\end{aligned}$$

and the second time:

$$\begin{aligned}&\int_0^{x_0}\int_0^{2\pi\sin(\gamma)}\left(L_{11}(F_1)+L_{12}(\mathrm{w})+L_{13}(F_1,w)\right)\mathrm{e}^x\sin^2(B_1x)\,d\varphi dx\\&=\int_0^{x_0}\int_0^{2\pi\sin(\gamma)} I_0\frac{\partial^2 w}{\partial t^2}\mathrm{e}^x\sin^2(B_1x)\,d\varphi dx.\end{aligned}$$

After the above processes, several transformation steps and renaming, we receive the equation systems to analyze the nonlinear dynamic problems of ES-FGM truncated conical shells in thermal environments:

$$l_{12}G^2f(t)^3+l_{14}f(t)^3+l_{16}Gf(t)^2+(l_{191}T_{\text{force}}+l_{192})f(t)=l_{21}\frac{\partial^2 f(t)}{\partial t^2},\tag{21a}$$
$$\begin{aligned}&l_{11}G^3f(t)^3+l_{13}Gf(t)^3+l_{15}G^2f(t)^2+l_{17}f(t)^2+(l_{181}T_{\text{force}}+l_{182})Gf(t)+l_{20}T_{\text{force}}+l_{24}q=\\&=l_{22}G\frac{\partial^2 f(t)}{\partial t^2},\end{aligned}\tag{21b}$$

in which l_{ijk} are coefficients denoted in Appendix C.

To solve the problem in this case, we will find the relationship between the nonlinear component $f(t)$ and the linear component G.

First, consider the case when the right hand side of Eqs. (21a) and (21b) is zero, in other words, consider the problem in the static case

$$l_{12}G^2f(t)^2+l_{14}f(t)^2+l_{16}Gf(t)+l_{191}T_{\text{force}}+l_{192}=0,\tag{22a}$$
$$l_{11}G^3f(t)^3+l_{13}Gf(t)^3+l_{15}G^2f(t)^2+l_{17}f(t)^2+(l_{181}T_{\text{force}}+l_{182})Gf(t)+l_{20}T_{\text{force}}=0.\tag{22b}$$

Substituting T_{force} from Eq. (22a) into Eq. (22b) yields

$$\begin{aligned}&\left(l_{11}-\tfrac{l_{12}l_{181}}{l_{191}}\right)(f(t))^3G^3+\left(l_{15}-\tfrac{l_{16}l_{181}}{l_{191}}-\tfrac{l_{20}l_{12}}{l_{191}}\right)(f(t))^2G^2+\left(l_{17}-\tfrac{l_{20}l_{14}}{l_{191}}\right)(f(t))^2\\&+\left(l_{13}-\tfrac{l_{14}l_{181}}{l_{191}}\right)G(f(t))^3+\left(-\tfrac{l_{192}l_{181}}{l_{191}}+l_{182}-\tfrac{l_{20}l_{16}}{l_{191}}\right)Gf(t)-\tfrac{l_{20}l_{192}}{l_{191}}=0.\end{aligned}\tag{23}$$

Equations (22a) and (22b) are quadratic equation and algebraic equation, respectively depending on f and G. Assuming that the relation between G and $f(t)$ could be defined as below [21]

$$G=\lambda f(t),\tag{24}$$

where λ is constant parameter that could be solved by substituting expression (24) into Eq. (23).

$$\lambda=\frac{-(l_{191}l_{17}-l_{20}l_{14})}{(-l_{192}l_{181}+l_{191}l_{182}-l_{20}l_{16})}.\tag{25}$$

From (22a), (22b) and after several transformations, we obtain the expression containing the linear natural frequency:

$$\frac{\partial^2 f(t)}{\partial t^2}=\omega_L^2\left(f(t)+\lambda_{11}f(t)^3+\lambda_{22}f(t)^5\right),\tag{26}$$

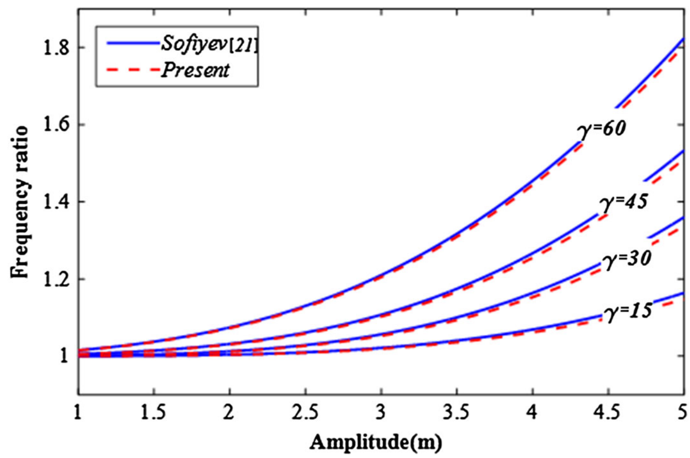

Fig. 2 The comparison with results in Sofiyev [21]

Table 1 Temperature-dependent coefficients of the constituent materials of the considered FGM truncated conical shells

Material	Properties	P_0	P_{-1}	P_1	P_2	P_3
Si_3N_4 (ceramic)	E (Pa)	384.43e9	0	− 3.07e−4	2.160e−7	− 8.946e-11
	ρ(kg/m^3)	2370	0	0	0	0
	α(K^{-1})	5.8723e−6	0	9.095e−4	0	0
	ν	0.24	0	0	0	0
SUS304 (metal)	E(Pa)	201.04e9	0	3.079e−4	− 6.534e−7	0
	ρ(kg/m^3)	8166	0	0	0	0
	α(K^{-1})	12.330e−6	0	8.086e−4	0	0
	ν	0.3177	0	0	0	0

where $\omega_L = \sqrt{-(l_{191}T_{\text{force}} + l_{192})/l_{21}}$ is the linear natural frequency and

$$\lambda_{11} = (l_{14} + l_{16}\lambda)\,/(l_{191}T_{\text{force}} + l_{192}), \quad \lambda_{22} = l_{12}\lambda^2/(l_{191}T_{\text{force}} + l_{192}).$$

Equation (26) is a second-order differential equation depending on the time variable. There are some methods to solve this equation such as the Galerkin method, Harmonic balance method, Rayleigh–Ritz method. In this paper, the Harmonic balance method is used [21]. In particular, by multiplying both sides of Eq. (26) with $\sin(\omega t)$, then integrating both sides from $0 \le t \le \frac{2\pi}{\omega}$, we receive the equation describing the oscillation amplitude and the nonlinear natural frequency of the shell:

$$\omega_{NL}^2 = \frac{1}{8}\omega_L^2\left(5A^4\lambda_{22} + 6A^2\lambda_{11} + 8\right). \tag{27}$$

4 Result and discussion

4.1 Validation

To validate the present formulation, the relation of amplitude and frequency in this study is considered with the results in Sofiyev [21] in case $\alpha = 15°, 30°, 45°, 60°$ without stiffened, foundation and no effect of thermal environment. The geometric parameters are chosen as follows: $R_1/h = 500$, $L/R_1 = 2$, $N = 1$, $\Delta T = 0$, $n_r = 0$. It is clear that there is no significant difference between two studies with the same method.

4.2 Nonlinear dynamic response of truncated conical shell

In this section, the paper concentrates on the nonlinear dynamic response analysis of eccentrically stiffened functionally graded truncated conical shells in thermal environments with the material property are assumed that depend on temperature and chosen as Table 1.

Table 2 The influence of temperature dependence of material properties and semi-vertex angle γ on the natural frequency

	$\gamma = 20$	$\gamma = 30$	$\gamma = 40$	$\gamma = 50$	$\gamma = 60$	$\gamma = 70$
TD	1.2654	0.8533	0.6416	0.514	0.432	0.3797
TID	1.2842	0.8664	0.6518	0.5226	0.4396	0.3867

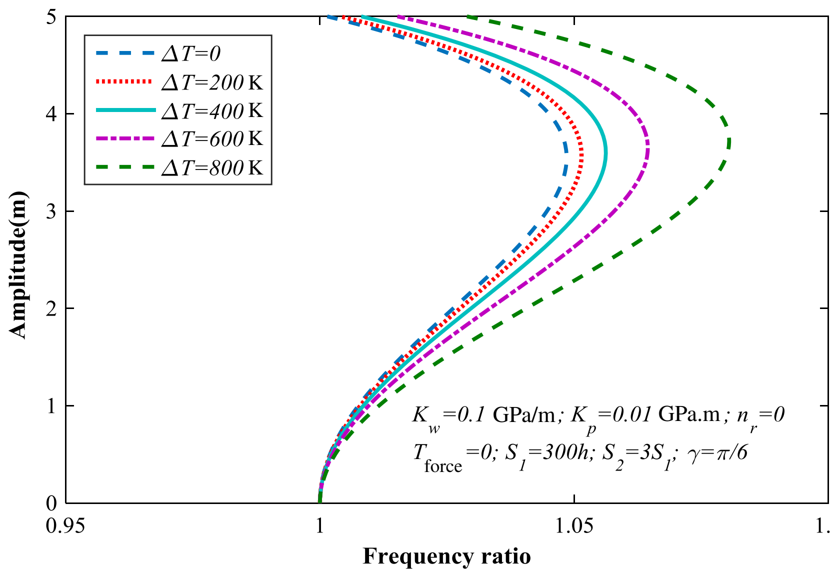

Fig. 3 The effect of environmental temperature variation on the relationship between vibration amplitude and frequency ratio

Table 2 describes the effect of the semi-vertex angle γ and temperature dependence of material properties on the natural frequency of the FGM truncated conical shell. It can be observed that the value of the natural frequency decreases when we increase the value of the angle γ. Moreover, the table also shows the significant difference in temperature dependent (TD) and temperature independent (TID), it means that the temperature dependence of material properties has a negative influence on the shell. $S_1/h = 300$, $S_2/S_1 = 3$, $n_r = 0$, $\Delta T = 0$, $k_w = 0.6$ GPa/m, $k_p = 0.06$ GPa.m

To analyze the effect of environmental temperature variation on the relationship between vibration amplitude and frequency ratio in the nonlinear state, the paper considers five values of temperature variation and the geometric parameters are chosen as shown in Fig 3. It is easy to recognize that, with the same amplitude value, the frequency ratio increases as the temperature variation increases.

Figure 4 shows the influences of the semi-vertex angle value on the frequency amplitude curves, and the figure indicates that the shell slope has a notable effect on the relationship between variation amplitude and frequency ratio. The increase in values of the angle γ from $\frac{\pi}{12}$ to $\frac{\pi}{6}$ leads to an increase in the frequency ratio.

Figure 5 describes the impact of the stiffeners and the axial force T_{force} on the nonlinear vibration behavior. The graph shows the remarkable effect of the stiffeners on the relationship between amplitude and frequency ratio. It is comprehensible that the vibration frequency of the stiffened shell is smaller than the frequency of the unstiffened shell. The figure also demonstrates that the axial force causes a noteworthy reduction in the frequency ratio with the same amplitude.

The effects of the elastic Winkler and Pasternak foundation on the dynamic response are presented in Fig. 6 and Fig. 7, respectively. Both figures indicate that increasing the values of elastic foundation such as K_{w} or K_{p} causes decreasing values of the vibration amplitude. Furthermore, the influence of the Pasternak foundation is greater than of the Winkler foundation.

Figure 8 shows how the dynamic response of the truncated conical shell is impacted by the power-law volume index N. $N = 0$ is the full ceramic shell and when we raise the value of N, this causes an increase in the metal ingredient and an decrease in the ceramic ingredient.

From the above-obtained figures, it could be ascertained that the vibration amplitude reduces when we raise the value of the power-law volume index N.

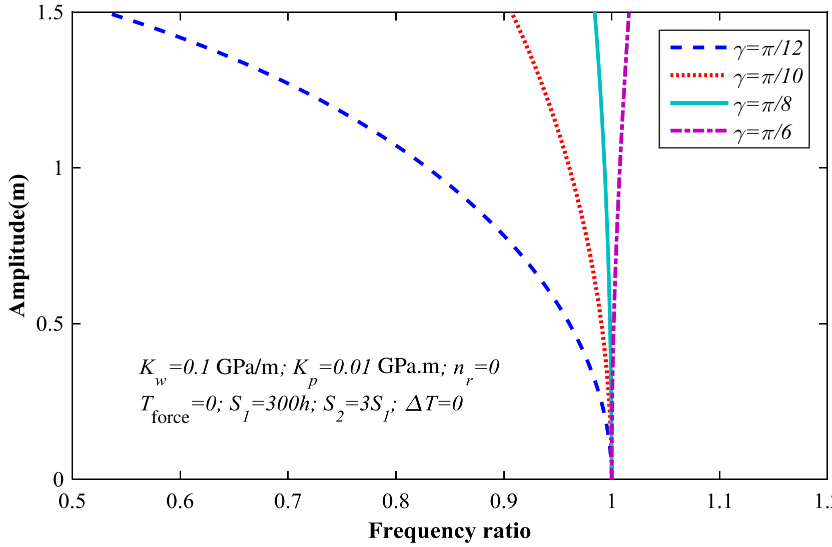

Fig. 4 The influences of the semi-vertex angle value on the relationship between vibration amplitude and frequency ratio in the nonlinear state

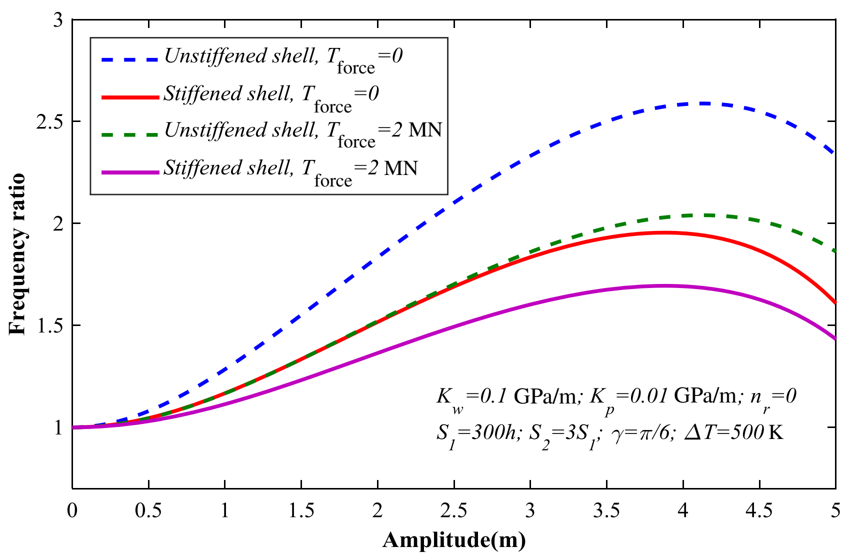

Fig. 5 The impact of stiffeners and axial force T_{force} on the nonlinear vibration behavior

The amplitude of deflection is discovered to be dependent on the value of the force amplitude Q in Fig. 9, they decrease when decreasing the values of the force amplitude, and the effect of the change of the force amplitude on the deflection is not small.

The evidence of the impact of the geometric parameters on the deflection–time curves is illustrated in Fig. 10, with the parameter chosen to be considered the length–thickness ratio L/h. In the case $L/h = 35$, the shell has the smallest deflection amplitude compared to the cases $L/h = 40$, $L/h = 45$, and the higher the ratio L/h, the greater the shell deflection, which means that when the shell becomes thinner, the deflection amplitude of the shell becomes bigger.

5 Concluding remarks

The present paper aims to study the vibration and nonlinear dynamic response of ES-FGM truncated conical shells on elastic foundations in thermal environments. The truncated conical shells are reinforced by ring stiffeners made of full metal or full ceramic depending on if the stiffeners are situated at the metal-rich or

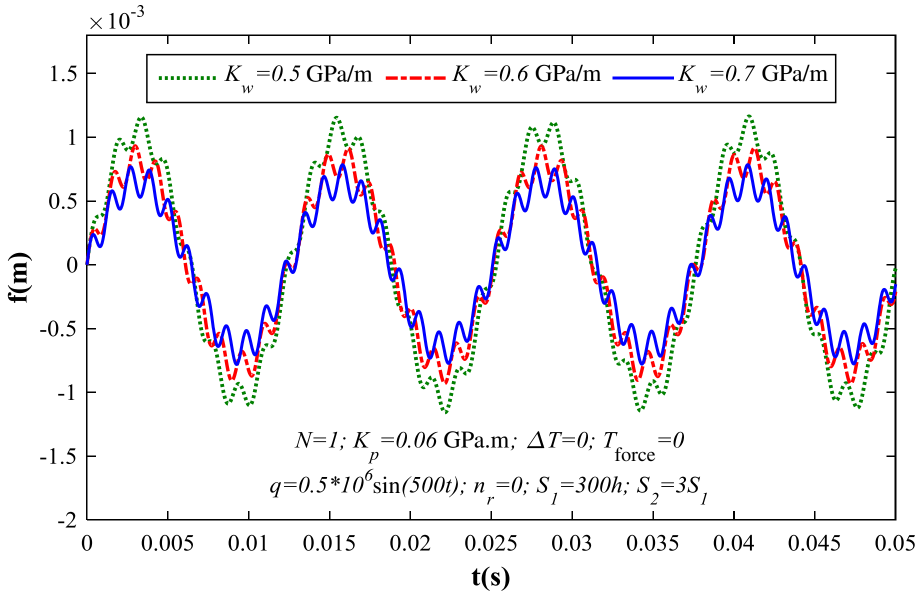

Fig. 6 The effects of the elastic Winkler foundation K_w on the dynamic response

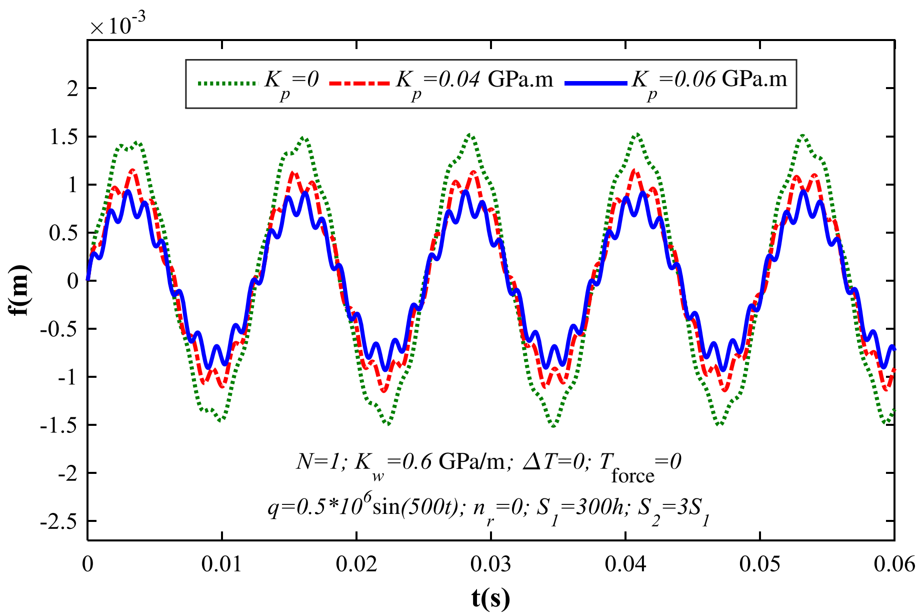

Fig. 7 The effects of the elastic Pasternak foundation K_p on the dynamic response

ceramic-rich side of the shell, respectively. In addition, the study not only assumes that the material properties depend on the environmental temperature variation, but also considers the thermal stresses in the stiffeners. Approximate solutions are assumed to satisfy the simply supported boundary condition, and the Galerkin method is applied to obtain closed-form relations of bifurcation type of the nonlinear response. From these expressions, the nonlinear dynamic response of eccentrically stiffened FGM truncated conical shells is analyzed and the results are illustrated in graphical form. The results show that the nonlinear dynamic response of ES-FGM truncated conical shells is greatly affected and influenced by the inhomogeneous dimensional parameters, stiffeners, temperatures and elastic foundations.

The numerical results support the following conclusions:

(i) The value of the natural frequency ES-FGM truncated conical shell decreases when the value of the semi-vertex angle γ of the cone increases.

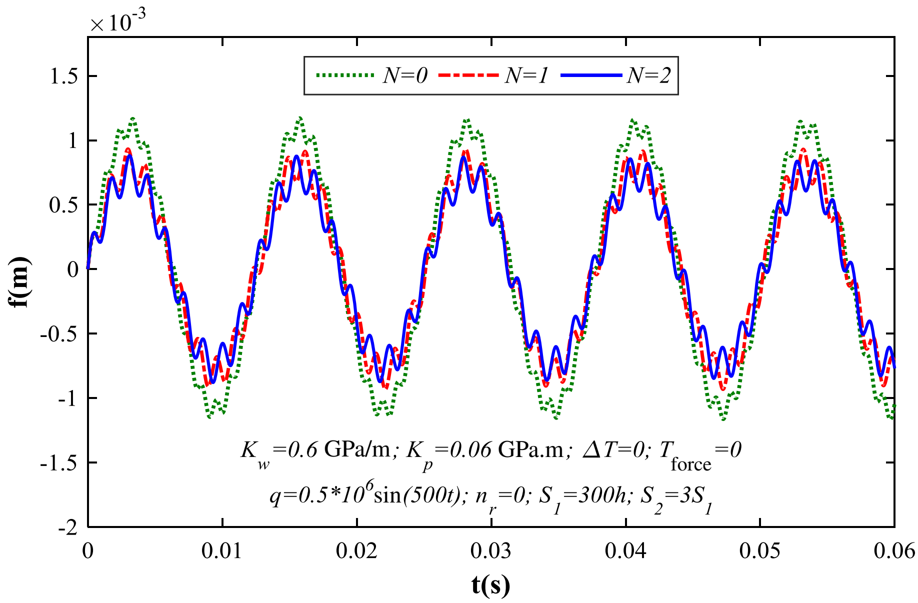

Fig. 8 The impact of the power-law volume index on deflection–time curves

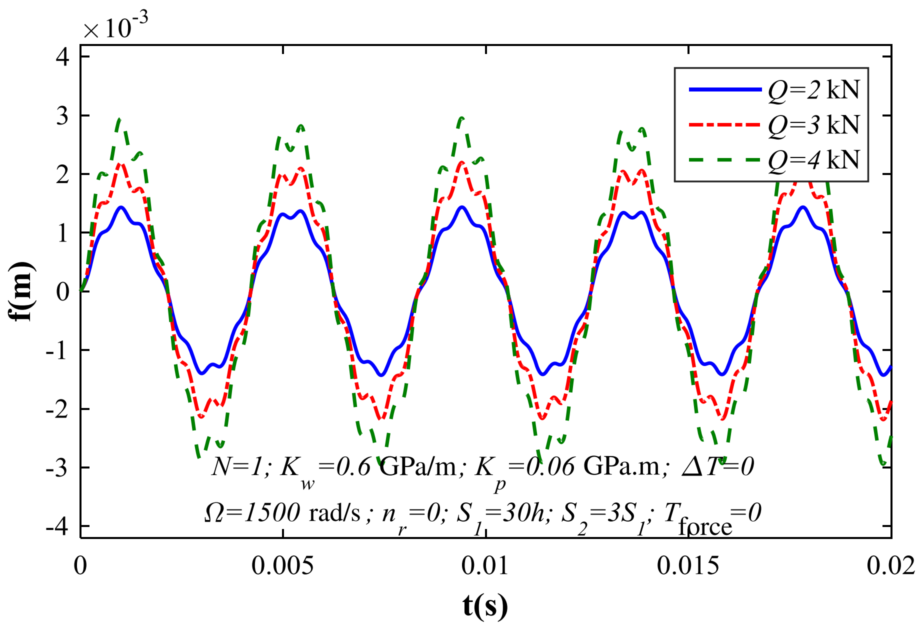

Fig. 9 The influence of the amplitude of external force Q on the dynamic response

(ii) The vibration frequency of the stiffened FGM truncated conical shell smaller than the frequency of unstiffened shell and the axial force causes a noteworthy reduction in the frequency ratio with the same amplitude.

(iii) In the case where amplitude value is constant, the frequency ratio increases when the temperature variation increases.

(iv) As the elastic foundation coefficients increase, the values of vibration amplitude of the ES-FGM truncated conical shell decrease. Furthermore, the influence of Pasternak foundation is bigger than Winkler foundation.

(v) The vibration amplitude of the ES-FGM truncated conical shell decreases when the value of the power-law volume index N increases.

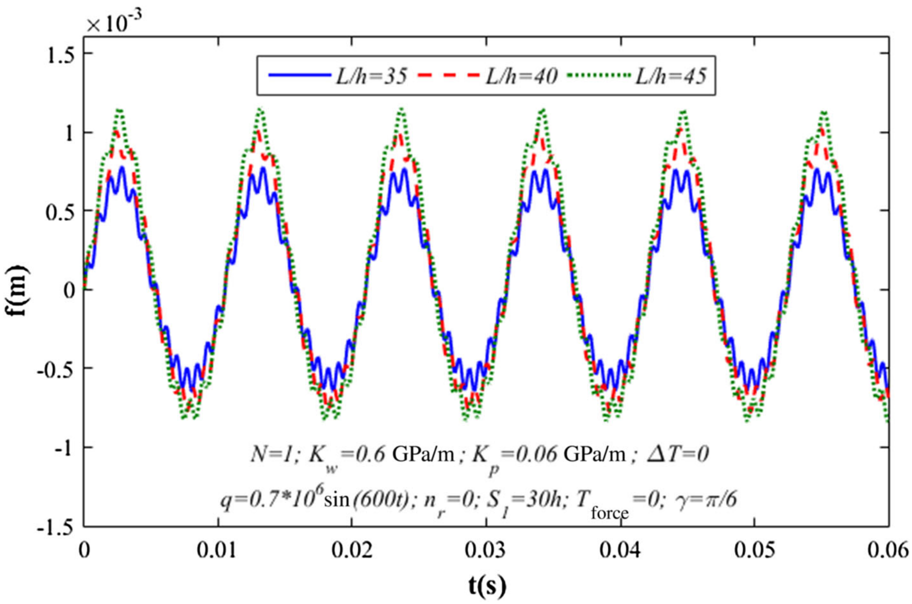

Fig. 10 The effect of geometry parameter ratio L/h on deflection–time curves

(vi) The deflection amplitude of an ES-FGM truncated conical shell increases when L/h increases. In other words, in the case where L is constant, the thinner the shells, the bigger the deflection amplitude of the shell.

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Compliance with ethical standards

Conflict of interest: The authors declare no conflict of interest.

Appendix A

$$L_{11}(F_1) = \frac{-2\cot(Y)}{S_1^2}F_1 + \left[\frac{-3e^x}{S_1^2}\cot(Y) + h_{111}\right]\frac{1}{e^x}\frac{\partial F_1}{\partial x} + \left[\frac{-e^x}{S_1^2}\cot(Y) + h_{112}\right]\frac{1}{e^x}\frac{\partial^2 F_1}{\partial x^2}$$
$$+ h_{113}\frac{1}{e^x}\frac{\partial^3 F_1}{\partial x^3} + h_{114}\frac{1}{e^x}\frac{\partial^4 F_1}{\partial x^4} + h_{115}\frac{1}{e^x}\frac{\partial^2 F_1}{\partial \phi^2}$$
$$+ h_{116}\frac{1}{e^x}\frac{\partial^3 F_1}{\partial x \partial \phi^2} + h_{117}\frac{1}{e^x}\frac{\partial^4 F_1}{\partial x^2 \partial \phi^2} + h_{118}\frac{1}{e^x}\frac{\partial^4 F_1}{\partial \phi^4},$$

$$L_{12}(\mathrm{w}) = -S_1 e^x K_w w + h_{121}\frac{1}{e^{3x}}\frac{\partial w}{\partial x} + \left[\frac{K_p e^{2x}}{S_1} + h_{122}\right]\frac{1}{e^{3x}}\frac{\partial^2 w}{\partial x^2} + h_{123}\frac{1}{e^{3x}}\frac{\partial^3 w}{\partial x^3} + h_{124}\frac{1}{e^{3x}}\frac{\partial^4 w}{\partial x^4}$$
$$+ \left[\frac{K_p e^{2x}}{S_1} + h_{125}\right]\frac{1}{e^{3x}}\frac{\partial^2 w}{\partial \phi^2} + h_{126}\frac{1}{e^{3x}}\frac{\partial^3 w}{\partial x \partial \phi^2} + h_{127}\frac{1}{e^{3x}}\frac{\partial^4 w}{\partial x^2 \partial \phi^2} + h_{128}\frac{1}{e^{3x}}\frac{\partial^4 w}{\partial \phi^4},$$

$$L_{13}(F_1, w) = \frac{2h_{13}}{e^x}F_1\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \phi^2}\right) + \frac{h_{13}}{e^x}\frac{\partial F_1}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial w}{\partial x} + 3\frac{\partial^2 w}{\partial \phi^2}\right) + \frac{h_{13}}{e^x}\frac{\partial^2 F_1}{\partial x^2}\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial \phi^2}\right)$$
$$+ 2h_{13}\frac{1}{e^x}\frac{\partial F_1}{\partial \phi}\left(\frac{\partial w}{\partial \phi} - \frac{\partial^2 w}{\partial x \partial \phi}\right) + h_{13}\frac{1}{e^x}\frac{\partial^2 F_1}{\partial \phi^2}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x}\right)$$

$$+ 2h_{13}\frac{1}{e^{x}}\frac{\partial^2 F_1}{\partial x\partial\phi}\left(\frac{\partial w}{\partial\phi}-\frac{\partial^2 w}{\partial x\partial\phi}\right),$$

$$L_{21}\left(F_1\right)=\frac{h_{211}}{e^{2x}}\frac{\partial F_1}{\partial x}+\frac{h_{212}}{e^{2x}}\frac{\partial^2 F_1}{\partial x^2}+\frac{h_{213}}{e^{2x}}\frac{\partial^3 F_1}{\partial x^3}+\frac{h_{214}}{e^{2x}}\frac{\partial^4 F_1}{\partial x^4}+\frac{h_{215}}{e^{2x}}\frac{\partial^2 F_1}{\partial\phi^2}+\frac{h_{216}}{e^{2x}}\frac{\partial^4 F_1}{\partial\phi^4}$$
$$+\frac{h_{217}}{e^{2x}}\frac{\partial^3 F_1}{\partial x\partial\phi^2}+\frac{h_{218}}{e^{2x}}\frac{\partial^4 F_1}{\partial x^2\partial\phi^2},$$

$$L_{22}\left(w\right)=\left(\frac{-e^{x}}{S_1^3}\cot(Y)+h_{221}\right)\frac{1}{e^{4x}}\frac{\partial w}{\partial x}+\left(\frac{e^{x}}{S_1^3}\cot(Y)+h_{222}\right)\frac{1}{e^{4x}}\frac{\partial^2 w}{\partial x^2}+\frac{h_{223}}{e^{4x}}\frac{\partial^3 w}{\partial x^3}+\frac{h_{224}}{e^{4x}}\frac{\partial^4 w}{\partial x^4}+$$
$$+\frac{h_{225}}{e^{4x}}\frac{\partial^2 w}{\partial\phi^2}+\frac{h_{226}}{e^{4x}}\frac{\partial^4 w}{\partial\phi^4}+\frac{h_{227}}{e^{4x}}\frac{\partial^3 w}{\partial x\partial\phi^2}+\frac{h_{228}}{e^{4x}}\frac{\partial^4 w}{\partial x^2\partial\phi^2},$$

$$L_{23}\left(w,w\right)=\frac{h_{231}}{e^{4x}}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial^2 w}{\partial x\partial\phi}\right)^2+\left(\frac{\partial w}{\partial\phi}\right)^2\right]+\frac{h_{232}}{e^{4x}}\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial\phi^2}\right)$$
$$+\frac{h_{233}}{e^{4x}}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial\phi^2}+\frac{h_{234}}{e^{4x}}\frac{\partial w}{\partial\phi}\frac{\partial^2 w}{\partial x\partial\phi},$$

with $h_{ijk}\ (i,\ j,k=1,2,3...)$ defined by

$$h_{111}=\frac{-\left(2C_{22}^*-2C_{11}^*+2C_{21}^*-2C_{12}^*\right)}{S_1^3};\ h_{112}=\frac{-\left(C_{22}^*-5C_{11}^*+3C_{21}^*-3C_{12}^*\right)}{S_1^3};$$

$$h_{113}=\frac{-\left(-4C_{11}^*+C_{21}^*-C_{12}^*\right)}{S_1^3};\ h_{114}=\frac{C_{11}^*}{S_1^3};$$

$$h_{115}=\frac{-\left(-2C_{22}^*+2C_{33}^*-2C_{21}^*\right)}{S_1^3};\ h_{116}=\frac{-\left(-3C_{21}^*-C_{12}^*+4C_{33}^*\right)}{S_1^3};$$

$$h_{117}=\frac{-\left(-C_{21}^*-C_{12}^*+2C_{33}^*\right)}{S_1^3};\ h_{118}=\frac{C_{22}^*}{S_1^3};\ h_{121}=\frac{-\left(-2D_{11}^*-2D_{21}^*+2D_{12}^*+2D_{22}^*\right)}{S_1^3};$$

$$h_{122}=\frac{-\left(5D_{11}^*+3D_{21}^*-3D_{12}^*-D_{22}^*\right)}{S_1^3};$$

$$h_{123}=\frac{-\left(-4D_{11}^*-D_{21}^*+D_{12}^*\right)}{S_1^3};\ h_{124}=\frac{-D_{11}^*}{S_1^3};\ h_{125}=\frac{-\left(2D_{12}^*+2D_{33}^*+2D_{22}^*\right)}{S_1^3};$$

$$h_{126}=\frac{-\left(-D_{21}^*-3D_{12}^*-4D_{33}^*\right)}{S_1^3};\ h_{127}=\frac{-\left(D_{21}^*+D_{12}^*+2D_{33}^*\right)}{S_1^3};\ h_{128}=\frac{-D_{22}^*}{S_1^3};\ h_{13}=\frac{1}{S_1^3};$$

$$h_{211}=\frac{2\left(A_{11}^*+A_{12}^*-A_{21}^*-A_{22}^*\right)}{S_1^4};\ h_{212}=\frac{\left(5A_{11}^*+3A_{12}^*-3A_{21}^*-A_{22}^*\right)}{S_1^4};\ h_{213}=\frac{\left(4A_{11}^*+A_{12}^*-A_{21}^*\right)}{S_1^4};$$

$$h_{214}=\frac{A_{11}^*}{S_1^4};\ h_{215}=\frac{\left(2A_{21}^*+2A_{22}^*+A_{33}^*\right)}{S_1^4};\ h_{216}=\frac{A_{22}^*}{S_1^4};\ h_{217}=\frac{\left(A_{12}^*+3A_{21}^*+2A_{33}^*\right)}{S_1^4};$$

$$h_{218}=\frac{\left(A_{12}^*+A_{21}^*+A_{33}^*\right)}{S_1^4};\ h_{221}=\frac{\left(2B_{11}^*-2B_{12}^*+2B_{21}^*-2B_{22}^*\right)}{S_1^4};\ h_{222}=\frac{\left(B_{22}^*-5B_{11}^*+3B_{12}^*-3B_{21}^*\right)}{S_1^4}$$

$$h_{223}=\frac{\left(-B_{12}^*+B_{21}^*+4B_{11}^*\right)}{S_1^4};\ h_{224}=\frac{-B_{11}^*}{S_1^4};\ h_{225}=\frac{\left(-2B_{12}^*-2B_{22}^*+B_{33}^*\right)}{S_1^4};\ h_{226}=\frac{-B_{22}^*}{S_1^4};$$

$$h_{227}=\frac{\left(-2B_{33}^*+B_{21}^*+3B_{12}^*\right)}{S_1^4};\ h_{228}=\frac{\left(B_{33}^*-B_{21}^*-B_{12}^*\right)}{S_1^4};$$

$$h_{231}=\frac{-1}{S_1^4};\ h_{232}=h_{233}=-h_{231};\ h_{234}=-2h_{231};$$

Appendix B

$$a_1 = 64 B_1{}^3 h_{213} - 4 B_1 h_{211};\quad a_2 = 256 B_1{}^4 h_{214} - 16 B_1{}^2 h_{212};$$
$$a_3 = -\left({}^1\!/_4\right) B_1 \left(3\ln^2(e)\, h_{232} - 4 B_1{}^2 h_{232} + 2\ln(e)\, h_{231}\right);$$
$$a_4 = \frac{\ln^3(e)\, h_{232}}{8} - \frac{3}{2}\ln(e)\, B_1{}^2 h_{232} + \frac{\ln^2(e)\, h_{231}}{8} - \frac{B_1{}^2 h_{231}}{2};$$
$$a_5 = \frac{1}{64 h_{216} B_2{}^4}\begin{pmatrix} -\ln^2(e)\, B_2{}^2 h_{231} - \ln^2(e)\, B_2{}^2 h_{233} - B_1{}^2 B_2{}^2 h_{231} + B_1{}^2 B_2{}^2 h_{233} + \ln^3(e)\, h_{232} \\ +\ln(e)\, B_1{}^2 h_{232} + \ln(e)\, B_2{}^2 h_{232} - \ln(e)\, B_2{}^2 h_{234} + \ln^2(e)\, h_{231} + B_1{}^2 h_{231} - B_2{}^2 h_{231} \end{pmatrix};$$
$$a_6 = \frac{27}{2} B_1{}^3 h_{213} + \frac{3}{2} B_1 B_2{}^2 h_{217} - \frac{3}{2} B_1 h_{211};$$
$$a_7 = \frac{81}{2} B_1{}^4 h_{214} + \frac{9}{2} B_1{}^2 B_2{}^2 h_{218} + \frac{h_{216} B_2{}^4}{2} - \frac{9}{2} B_1{}^2 h_{212};$$
$$a_8 = \frac{\ln^2(e)}{8} B_2{}^2 h_{233} - \frac{B_1{}^2 B_2{}^2 h_{233}}{2} - \frac{\ln^3(e)\, h_{232}}{4} + \frac{13}{8}\ln(e)\, B_1{}^2 h_{232} - \frac{\ln(e)\, B_2{}^2 h_{232}}{8} - \frac{\ln^2(e)}{4} h_{231} + \frac{B_1{}^2 h_{231}}{2};$$
$$a_9 = -\frac{B_1}{8}\left(-4\ln(e)\, B_2{}^2 h_{233} + 9\ln^2(e)\, h_{232} - 6 B_1{}^2 h_{232} + 2 B_2{}^2 h_{232} + 6\ln(e)\, h_{231}\right);$$
$$a_{10} = 4 B_1{}^3 h_{213} + 4 B_1 B_2{}^2 h_{217} - B_1 h_{211};$$
$$a_{11} = 8 B_1{}^4 h_{214} + 8 B_1{}^2 B_2{}^2 h_{218} + 8 h_{216} B_2{}^4 - 2 B_1{}^2 h_{212};$$
$$a_{12} = \frac{\ln(e)}{4} B_1 B_2{}^2 h_{231} + \frac{\ln(e)}{4} B_1 B_2{}^2 h_{233} - \frac{3}{8}\ln^2(e)\, B_1 h_{232} + \frac{B_1{}^3 h_{232}}{8} - \frac{B_1 B_2{}^2 h_{232}}{8} + \frac{B_1 B_2{}^2 h_{234}}{8} - \frac{\ln(e)\, B_1 h_{231}}{4};$$
$$a_{13} = \frac{1}{8}\begin{pmatrix} \ln^3(e)\, h_{232} - \ln^2(e)\, B_2{}^2 (h_{231} + h_{233}) + \ln^2(e)\, h_{231} - 3\ln(e)\, B_1{}^2 h_{232} \\ +\ln(e)\, B_2{}^2 (h_{232} - h_{234}) + B_1{}^2 B_2{}^2 (h_{231} + h_{233}) - B_1{}^2 h_{231} - B_2{}^2 h_{231} \end{pmatrix};$$
$$a_{14} = -(1/2)\, B_1 \left(B_1{}^2 h_{213} - 4 B_1{}^2 h_{214} + B_2{}^2 h_{217} - 2 B_2{}^2 h_{218} - h_{211} + 2 h_{212} - 3 h_{213} + 4 h_{214}\right);$$
$$a_{15} = (1/2)\begin{pmatrix} B_1{}^4 h_{214} + B_1{}^2 B_2{}^2 h_{218} + h_{216} B_2{}^4 - B_1{}^2 h_{212} + 3 B_1{}^2 h_{213} \\ -6 B_1{}^2 h_{214} + B_2{}^2 h_{217} - B_2{}^2 h_{218} - h_{211} + h_{212} - h_{213} + h_{214} \end{pmatrix};$$
$$a_{16} = 1/2\, B_1 \left(\ln^3(e)\,(4 h_{224} + 3 h_{223}) - 2\ln(e)\left(2 B_1{}^2 h_{224} - B_2{}^2 h_{228}\right) - B_1{}^2 h_{223} - B_2{}^2 h_{227} + 2\ln(e)\, h_{222} + h_{221}\right);$$
$$a_{17} = \frac{1}{2}\begin{Bmatrix} \ln^4(e)\, h_{224} - 6\ln^2(e)\, B_1{}^2 h_{224} - \ln^2(e)\, B_2{}^2 h_{228} + B_1{}^4 h_{224} + B_1{}^2 B_2{}^2 h_{228} + B_2{}^4 h_{226} \\ +\ln^3(e) h_{223} - 3\ln(e) B_1{}^2 h_{223} - \ln(e) B_2{}^2 h_{227} + \ln^2(e) h_{222} - B_1{}^2 h_{222} - B_2{}^2 h_{225} + \ln(e) h_{221} \end{Bmatrix};$$
$$a_{18} = -(1/2)\, B_1 \left(B_1{}^2 h_{213} + B_2{}^2 h_{217} - h_{211}\right);$$
$$a_{19} = (1/2)\left(B_1{}^4 h_{214} + B_1{}^2 B_2{}^2 h_{218} + h_{216} B_2{}^4 - B_1{}^2 h_{212}\right);$$
$$a_{20} = (1/8)\, B_1 \left(-4\ln(e)\, B_2{}^2 h_{233} + 9\ln^2(e)\, h_{232} + 2 B_1{}^2 h_{232} + 2 B_2{}^2 h_{232} + 6\ln(e)\, h_{231}\right);$$
$$a_{21} = \frac{1}{2} B_2{}^2 h_{233}\left(-\frac{3}{4}\ln^2(e) + B_1{}^2\right) + \frac{3}{4}\ln^3(e)\, h_{232} + \frac{1}{8}\ln(e)\, h_{232}\left(B_1{}^2 + 3 B_2{}^2\right) + \frac{3}{4}\ln^2(e)\, h_{231} + \frac{1}{2} B_1{}^2 h_{231};$$
$$a_{22} = -(1/2)\, B_1 \left(B_1{}^2 h_{213} + B_2{}^2 h_{217} - h_{211}\right);$$
$$a_{23} = (1/2)\left(B_1{}^4 h_{214} + B_1{}^2 B_2{}^2 h_{218} + h_{216} B_2{}^4 - B_1{}^2 h_{212}\right);$$
$$a_{24} = \frac{B_1 \cos(\gamma)\,(2\ln(e) - 1)}{2 S_1{}^3};\quad a_{25} = \frac{e^{-2x}\cos(\gamma)\, f(t)\left(\ln^2(e) - B_1{}^2 - \ln(e)\right)}{2 S_1{}^3};$$

$$a_{26} = 2B_1\left(4{B_1}^2h_{213} - 16{B_1}^2h_{214} - h_{211} + 2h_{212} - 3h_{213} + 4h_{214}\right);$$
$$a_{27} = 16{B_1}^4h_{214} - 4{B_1}^2h_{212} + 12{B_1}^2h_{213} - 24{B_1}^2h_{214} - h_{211} + h_{212} - h_{213} + h_{214};$$
$$a_{28} = B_1\left(4\ln^3(e)\,h_{224} - 16\ln(e)\,B_1^2h_{224} + 3\ln^2(e)\,h_{223} - 4B_1^2h_{223} + 2\ln(e)\,h_{222} + h_{221}\right);$$
$$a_{29} = \frac{1}{2}\left\{\begin{array}{l} -h_{224}\ln^4(e) + 8B_1^2h_{224}\left(3\ln^2(e) - 2B_1^2\right) - h_{223}\ln^3(e) \\ +12\ln(e)\,B_1^2h_{223} - h_{222}\ln^2(e) + 2B_1^2h_{222} - h_{221}\ln(e) \end{array}\right\};\ a_{30} = 2B_1\left(4B_1^2h_{213} - h_{211}\right);$$
$$a_{31} = 4B_1^2\left(4B_1^2h_{214} - h_{212}\right);\ a_{32} = (1/2)\ \ln(e)\,B_1\left(3\ln(e)\,h_{232} + 2h_{231}\right);$$
$$a_{33} = -(1/2)\ \ln(e)\left(\ln^2(e)\,h_{232} - 2B_1^2h_{232} + \ln(e)\,h_{231}\right);\ a_{34} = a_{30};\ a_{35} = a_{31};$$
$$a_{36} = (1/4)\ B_1\left(2\ln(e)\,B_2^2h_{231} - 2\ln(e)\,B_2^2h_{233} + 3\ln^2(e)\,h_{232} - B_1^2h_{232}\right.$$
$$\left. + B_2^2h_{232} + B_2^2h_{234} + 2\ln(e)\,h_{231}\right);$$
$$a_{37} = \frac{1}{4}\left\{\begin{array}{l} \ln^2(e)\left(-B_2^2h_{231} + B_2^2h_{233} - \ln(e)\,h_{232} - h_{231}\right) + B_1^2h_{231} - B_2^2h_{231} \\ +3\ln(e)\left(3B_1^2h_{232} - B_2^2h_{232} - B_2^2h_{234}\right) + B_1^2B_2^2h_{231} - B_1^2B_2^2h_{233} \end{array}\right\};\ a_{38} = a_{30};\ a_{39} = 2a_{31};$$
$$a_{40} = a_{24};\ a_{41} = -\frac{\cos(\gamma)\left(\ln^2(e) - 4B_1^2 - \ln(e)\right)}{2S_1^3};$$
$$a_{42} = -h_{211} + h_{212} - h_{213} + h_{214};\ a_{43} = (1/2)\ln(e)\left(\ln^3(e)\,h_{224} + \ln^2(e)\,h_{223} + \ln(e)\,h_{222} + h_{221}\right).$$

Appendix C

$$l_{11} = \frac{16h_{13}\pi^5\sin(\gamma)\,m^4\left(e^{x_0}-1\right)}{\left(x_0^2+36m^2\pi^2\right)\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+4m^2\pi^2\right)}\left\{\begin{array}{l} \pi m x_0^2\left(-11x_0^2+4m^2\pi^2\right)K_{61} \\ -x_0\left(48m^4\pi^4+32\pi^2x_0^2m^2-x_0^4\right)K_{51} \end{array}\right\}$$
$$+\frac{16h_{13}\pi^5\sin(\gamma)\,m^4\left(e^{x_0}-1\right)}{\left(x_0^2+36m^2\pi^2\right)\left(x_0^2+64m^2\pi^2\right)\left(x_0^2+16m^2\pi^2\right)}\left\{\begin{array}{l} -x_0\left(1728m^4\pi^4+152\pi^2x_0^2m^2-x_0^4\right)K_{13} \\ +2\pi m\left(2304m^4\pi^4+88\pi^2x_0^2m^2-11x_0^4\right)K_{14} \end{array}\right\};$$
$$l_{12} = \frac{4\left(e^{x_0}-1\right)m^2\pi^3h_{13}}{x_0^2\sin(\gamma)\left(x_0^2+4m^2\pi^2\right)\left(x_0^2+16m^2\pi^2\right)}\left\{\begin{array}{l} m^2\pi^2\left(\cos^2(\gamma)-1\right)\left(5m^2\pi^2+2x_0^2\right)\left(2K_{51}+K_{101}\right) \\ -n^2\left(x_0^2+4m^2\pi^2\right)^2\left(K_{51}-2m^2\pi^2K_{101}\right) \end{array}\right\}$$
$$+\frac{2\left(e^{x_0}-1\right)m^3\pi^4h_{13}\left\{\begin{array}{l} 3m\pi\left(x_0^2+4m^2\pi^2\right)\left(\cos^2(\gamma)-1\right)\left\{2x_0\left(9m^2\pi^2-x_0^2\right)K_{12}-m\pi\left(17x_0^2+72m^2\pi^2\right)K_{11}\right\} \\ -n^2\left(12m^2\pi^2+x_0^2\right)\left\{2\left(4m^2\pi^2-x_0^2\right)K_{12}+x_0^2\left(-x_0^2+44m^2\pi^2\right)K_{11}\right\} \end{array}\right\}}{x_0^3\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+36m^2\pi^2\right)\left(x_0^2+4m^2\pi^2\right)\sin(\gamma)}$$
$$+\frac{4\left(e^{x_0}-1\right)m^2\pi^3h_{13}\left\{\begin{array}{l} m\pi\left(\cos^2(\gamma)-1\right)\left\{m\pi\left(168m^2\pi^2+13x_0^2\right)K_{13}+2\left(11m^2\pi^2+x_0^2\right)x_0K_{14}\right\} \\ -n^2x_0\left(x_0^2+16m^2\pi^2\right)\left\{x_0K_{13}-6m\pi K_{14}\right\} \end{array}\right\}}{x_0^2\sin(\gamma)\left(x_0^2+36m^2\pi^2\right)\left(x_0^2+16m^2\pi^2\right)}$$
$$+\frac{2\left(e^{x_0}-1\right)m^3\pi^4h_{13}\left\{\begin{array}{l} \left(\cos^2(\gamma)-1\right)\left\{2\left(8m^4\pi^4+3m^2\pi^2x_0^2+x_0^4\right)K_{61}-m^2\pi^2\left(8m^2\pi^2+11x_0^2\right)K_{91}\right\} \\ +n^2x_0^2\left(x_0^2+4m^2\pi^2\right)\left\{3K_{61}+K_{91}\right\} \end{array}\right\}}{\sin(\gamma)\left(x_0^2+4m^2\pi^2\right)x_0^3\left(x_0^2+16m^2\pi^2\right)};$$
$$l_{13} = -\frac{1}{64}\,\frac{h_{13}\left(e^{x_0}-1\right)\left(\cos^2(\gamma)\,s_{13}e_1+s_{13}e_2\right)}{n\left(x_0^2+16m^2\pi^2\right)x_0^3\left(x_0^2+4m^2\pi^2\right)\left(x_0^2+36m^2\pi^2\right)\sin(\gamma)};$$
$$s_{13}e_1 = \pi^4nm^3\left\{\begin{array}{l} 384\pi m x_0\left(7x_0^2+12m^2\pi^2\right)\left(x_0^2+6m^2\pi^2\right)K_{12}+1024\pi^6nm^5x_0^2\left(4m^2\pi^2-11x_0^2\right)K_{62} \\ +256\left(x_0^2+6m^2\pi^2\right)\left(72m^4\pi^4+26m^2\pi^2x_0^2-x_0^4\right)K_{11} \\ -1024\pi m x_0\left(-x_0^4+48m^4\pi^4+32m^2\pi^2x_0^2\right)K_{52} \\ -\left(x_0^2+36m^2\pi^2\right)\left(-x_0^2+2m^2\pi^2\right)\left(896\pi m x_0K_{101}-256\left(x_0^2+2m^2\pi^2\right)K_{91}\right) \end{array}\right\};$$
$$s_{13}e_2 = \pi^4nm^3\left\{\begin{array}{l} 384\pi m x_0\left(x_0^2+6m^2\pi^2\right)\left(32n^2m^2\pi^2-12m^2\pi^2-7x_0^2+8n^2x_0^2\right)K_{12} \\ -1024\pi^2m^2x_0^2\left(4m^2\pi^2-11x_0^2\right)K_{62}+1024\pi m x_0\left(-x_0^4+48m^4\pi^4+32m^2\pi^2x_0^2\right)K_{52} \\ -128\left(864m^6\pi^6+3n^2x_0^6+456m^4\pi^4x_0^2-48n^2m^4\pi^4x_0^2+40m^2\pi^2x_0^4-2x_0^6\right)K_{11} \\ -128\pi m x_0\left(8n^2-7\right)\left(x_0^2+36m^2\pi^2\right)\left(-x_0^2+2m^2\pi^2\right)K_{101} \\ -128\left(x_0^2+36m^2\pi^2\right)\left(8m^4\pi^4+12n^2m^2\pi^2x_0^2+3n^2x_0^4-2x_0^4\right)K_{91} \end{array}\right\};$$
$$l_{14} = -\frac{4h_{13}\pi^3m^2\left(e^{x_0}-1\right)\left(\begin{array}{l} K_7\left(n^2x_0^4-10m^4\pi^4+10n^2m^2\pi^2x_0^2-4m^2\pi^2x_0^2\right) \\ +K_{52}\left(32n^2m^4\pi^4+20m^4\pi^4+8m^2\pi^2x_0^2+2n^2x_0^4+16n^2m^2\pi^2x_0^2\right) \end{array}\right)}{x_0^2\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+4m^2\pi^2\right)\sin(\gamma)}$$

$$+\frac{4h_{13}\pi^4m^3\left(\mathrm{e}^{x_0}-1\right)\begin{pmatrix}K_8\left(8m^4\pi^4+x_0^4+3m^2\pi^2x_0^2+24n^2m^2\pi^2x_0^2+3n^2x_0^4\right)\\+K_{62}\left(-6m^2\pi^2x_0^2-2x_0^4-16m^4\pi^4+24n^2m^2\pi^2x_0^2+6n^2x_0^4\right)\end{pmatrix}}{\left(x_0^2+16m^2\pi^2\right)x_0^3\left(x_0^2+4m^2\pi^2\right)\sin(\gamma)}$$
$$-\frac{4K_{15}\left(\mathrm{e}^{x_0}-1\right)\left(2n^2m^2\pi^2-m^2\pi^2+n^2x_0^2\right)m^2\pi^3h_{13}}{x_0^2\sin(\gamma)\left(x_0^2+4m^2\pi^2\right)};$$

$$l_{15}=-\frac{\pi^2m\begin{pmatrix}8m^3\pi^3h_{114}K_{51}-4m^2\pi^2h_{113}K_{61}x_0\\-2m\pi K_{51}h_{112}x_0^2+K_{61}h_{111}x_0^3\end{pmatrix}\sin(\gamma)}{x_0^3}$$
$$+\frac{24m\pi^2h_{13}\left(2m\pi K_3-2m\pi K_{16}-K_4x_0\right)\sin(\gamma)}{48x_0}$$
$$-\frac{16\pi^5h_{13}m^4\left[K_{53}\left(48m^4\pi^4-x_0^4+32m^2\pi^2x_0^2\right)-m\pi K_{63}x_0\left(-11x_0^2+4m^2\pi^2\right)\right]\left(\mathrm{e}^{x_0}-1\right)\sin(\gamma)}{\left(x_0^2+4m^2\pi^2\right)x_0^2\left(x_0^2+36m^2\pi^2\right)\left(x_0^2+16m^2\pi^2\right)}$$
$$-\frac{\cos(\gamma)\,16\pi^3m^2\left(\mathrm{e}^{x_0}-1\right)\begin{Bmatrix}\left(x_0^2+36m^2\pi^2\right)\left[\left(8m^2\pi^2-x_0^2\right)\left(-x_0^2+2m^2\pi^2\right)+18x_0^2m^2\pi^2S_1^2\right]K_{51}\\-\left(x_0^2+36m^2\pi^2\right)\left[\left(-4m^2\pi^2+2x_0^2\right)+S_1^2\left(8m^2\pi^2-x_0^2\right)\right]3m\pi x_0K_{61}\\-\left[2\left(4m^2\pi^2-x_0^2\right)\left(8m^2\pi^2-x_0^2\right)+S_1^2x_0^2\left(44m^2\pi^2-x_0^2\right)\right]6m\pi x_0K_{14}\\+\left[\left(44m^2\pi^2-x_0^2\right)\left(8m^2\pi^2-x_0^2\right)-72m^2\pi^2S_1^2\left(4m^2\pi^2-x_0^2\right)\right]x_0^2K_{13}\end{Bmatrix}}{2S_1^2x_0^2\left(x_0^2+4m^2\pi^2\right)\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+36m^2\pi^2\right)};$$

$$l_{16}=\frac{\pi^2m\cos(\gamma)\left(\mathrm{e}^{x_0}-1\right)}{S_1^2\left(x_0^2+16m^2\pi^2\right)x_0^2\left(x_0^2+4m^2\pi^2\right)}\begin{Bmatrix}\left(x_0^2+16m^2\pi^2\right)\left(2m^2\pi^2+3S_1^2x_0^2-4x_0^2\right)\pi mK_{101}\\+\left(3S_1^2x_0^2-24m^2\pi^2S_1^2-4x_0^2+18m^2\pi^2\right)3\pi mx_0^2K_{12}\\+\left(54x_0^2m^2\pi^2S_1^2+2x_0^4-25x_0^2m^2\pi^2+72m^4\pi^4\right)x_0K_{11}\\+\left(x_0^2+16m^2\pi^2\right)\left(6m^2\pi^2S_1^2-m^2\pi^2+2x_0^2\right)x_0K_{91}\end{Bmatrix}$$
$$+\begin{pmatrix}\left(\pi n^2h_{115}x_0^4-n^2m^2\pi^3h_{117}x_0^2+2m^2\pi^3h_{112}x_0^2-2h_{114}m^4\pi^5\right)\cos^2(\gamma)-m^2\pi^3h_{112}x_0^2\cos^4(\gamma)\\+n^2m^2\pi^3h_{117}x_0^2+h_{114}m^4\pi^5-m^2\pi^3h_{112}x_0^2+\pi n^4h_{118}x_0^4-\pi n^2h_{115}x_0^4+m^2\pi^3h_{114}m^2\pi^2\cos^4(\gamma)\end{pmatrix}\frac{K_{101}}{2x_0^3\sin^3(\gamma)}$$
$$+\begin{pmatrix}m\pi^2\left(h_{113}m^2\pi^2x_0-h_{111}x_0^3\right)\cos^4(\gamma)-m\pi^2h_{111}x_0^3+h_{113}m^3\pi^4x_0+n^2m\pi^2h_{116}x_0^3\\+\left(2m\pi^2h_{111}x_0^3-n^2m\pi^2h_{116}x_0^3-2h_{113}m^3\pi^4x_0\right)\cos^2(\gamma)\end{pmatrix}\frac{K_{91}}{2x_0^3\sin^3(\gamma)}$$
$$-\frac{h_{13}m\pi^2\begin{pmatrix}2\cos^2(\gamma)m^2\pi^2-\cos^2(\gamma)x_0^2\\+2n^2x_0^2+x_0^2-2m^2\pi^2\end{pmatrix}K_2}{8x_0^2\sin(\gamma)}+\frac{h_{13}m\pi^2\begin{pmatrix}2\cos^2(\gamma)m^2\pi^2+\cos^2(\gamma)x_0^2\\+2n^2x_0^2-x_0^2-2m^2\pi^2\end{pmatrix}K_4}{4x_0^2\sin(\gamma)}$$
$$-\frac{h_{13}m^2\pi^3\left(\cos^2(\gamma)-1+4n^2\right)K_3}{4x_0\sin(\gamma)}+\frac{h_{13}m^2\pi^3\left(\cos^2(\gamma)-1\right)K_{16}}{2x_0\sin(\gamma)}$$
$$+\frac{h_{13}m^2\pi^3\left(3\cos^2(\gamma)-3+4n^2\right)K_1}{8x_0\sin(\gamma)}$$
$$+\frac{2h_{13}\pi^4m^3\left(\mathrm{e}^{x_0}-1\right)}{\left(x_0^2+4m^2\pi^2\right)x_0^3\left(x_0^2+16m^2\pi^2\right)\sin(\gamma)}\begin{Bmatrix}\left(\cos^2(\gamma)-1\right)\begin{bmatrix}\left(3m^2\pi^2x_0^2+x_0^4+8m^4\pi^4\right)2K_{63}\\-m^2\pi^2\left(8m^2\pi^2+11x_0^2\right)K_{92}\end{bmatrix}\\+\left(x_0^2+4m^2\pi^2\right)\left[6n^2x_0^2K_{63}-n^2x_0^2K_{92}\right]\end{Bmatrix}$$
$$+\frac{4h_{13}\pi^3m^2\left(\mathrm{e}^{x_0}-1\right)\begin{Bmatrix}\left(2m^2\pi^2\left(5m^2\pi^2+2x_0^2\right)\left(\cos^2(\gamma)-1\right)-n^2\left(x_0^2+4m^2\pi^2\right)^2\right)K_{53}\\+\left(\left(-x_0^2+5m^2\pi^2\right)\left(\cos^2(\gamma)-1\right)+2n^2\left(x_0^2+4m^2\pi^2\right)\right)\pi^2m^2K_{102}\end{Bmatrix}}{x_0^2\left(x_0^2+4m^2\pi^2\right)\left(x_0^2+16m^2\pi^2\right)\sin(\gamma)};$$

$$l_{17}=\frac{8\pi^3m^2\left(-1+\mathrm{e}^{x_0}\right)\cos(\gamma)}{x_0S_1^2\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+4m^2\pi^2\right)}\begin{Bmatrix}\left(2x_0^2-4m^2\pi^2-x_0^2S_1^2+8m^2\pi^2S_1^2\right)3\pi mK_{62}\\-\left(x_0^4-10m^2\pi^2x_0^2+16m^4\pi^4+18m^2\pi^2S_1^2x_0^2\right)K_{52}\end{Bmatrix}$$
$$+\frac{\sin(\gamma)}{x_0^3}\begin{pmatrix}-8m^4\pi^5K_{52}h_{114}+4x_0m^3\pi^4K_{62}h_{113}\\+2x_0^2m^2\pi^3K_{52}h_{112}-x_0^3m\pi^2K_{62}h_{111}\end{pmatrix}$$
$$+\frac{h_{13}m\pi^2\left(K_2x_0^2-m^2\pi^2K_2+2m\pi K_1x_0\right)\cos^2(\gamma)}{4x_0^2\sin(\gamma)}$$
$$-\frac{2h_{13}\pi^4m^3\left(2m^2\pi^2-x_0^2\right)\left(4K_{92}m^2\pi^2-7K_{102}x_0m\pi+2K_{92}x_0^2\right)\left(\mathrm{e}^{x_0}-1\right)\cos^2(\gamma)}{\left(x_0^2+4m^2\pi^2\right)x_0^3\left(x_0^2+16m^2\pi^2\right)\sin(\gamma)}$$
$$+\frac{h_{13}m\pi^2\left(-2m\pi K_1x_0-K_2x_0^2+m^2\pi^2K_2+2m\pi n^2K_1x_0+n^2K_2x_0^2\right)}{4x_0^2\sin(\gamma)}$$

$$+\frac{2h_{13}m^3\pi^4\left(\mathrm{e}^{x_0}-1\right)\begin{pmatrix}K_{102}m\pi x_0\left(16n^2m^2\pi^2-8n^2x_0^2+7x_0^2-14m^2\pi^2\right)\\+K_{92}\left(12n^2m^2\pi^2x_0^2+8m^4\pi^4+3n^2x_0^4-2x_0^4\right)\end{pmatrix}}{\left(x_0^2+4m^2\pi^2\right)x_0^3\left(x_0^2+16m^2\pi^2\right)\sin(\gamma)};$$

$$l_{181}=\frac{4\pi^3m^2\left(\mathrm{e}^{3x_0}-1\right)\sin(\gamma)\,h_{13}S_1^2\left(24\pi^2m^2\left(2B_1^2-1\right)+144\pi\,mx_0B_1-4B_1^2\left(4\pi^2m^2+9x_0^2\right)\right)}{3\left(16\pi^2m^2+9x_0^2\right)\left(4\pi^2m^2+9x_0^2\right)}$$
$$+\frac{4m^2\pi^3\left(\mathrm{e}^{x_0}-1\right)\sin(\gamma)}{\left(16\pi^2m^2+x_0^2\right)\left(4\pi^2m^2+x_0^2\right)}\begin{pmatrix}12\pi^2m^2\left(2B_1^2h_{13}S_1^2-h_{13}S_1^2\right)+12\pi\,mx_0B_1S_1^2h_{13}\\-8B_1^2S_1^2h_{13}\pi^2m^2-2B_1^2S_1^2h_{13}x_0^2\end{pmatrix};$$

$$l_{182}=\frac{\pi x_0\sin(\gamma)\,\cot(\gamma)}{S_1^2}$$
$$\left[3K_{16}\left(S_1^2-1\right)-\left(3K_3\left(S_1^2-1\right)-\frac{6m\pi}{x_0}K_4S_1^2+\frac{4m^2\pi^2}{x_0^2}K_3+\frac{4m\pi}{x_0}K_4\right)\right]$$
$$+\frac{24\pi^4m^3\left(2x_0^2-4m^2\pi^2+8m^2\pi^2S_1^2-S_1^2x_0^2\right)\left(\mathrm{e}^{x_0}-1\right)\cos(\gamma)\,K_{63}}{x_0S_1^2\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+4m^2\pi^2\right)}$$
$$-\frac{8\pi^3m^2\left(16m^4\pi^4-10m^2\pi^2x_0^2+x_0^4+18m^2\pi^2S_1^2x_0^2\right)\left(\mathrm{e}^{x_0}-1\right)\cos(\gamma)\,K_{53}}{x_0^2S_1^2\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+4m^2\pi^2\right)}$$
$$+\frac{4K_3\sin(\gamma)\begin{pmatrix}32h_{114}m^4\pi^4-8\pi^2m^2h_{112}x_0^2-12\pi^2m^2h_{114}x_0^2\\+12\pi^2m^2h_{113}x_0^2-h_{112}x_0^2+h_{111}x_0^4-h_{114}x_0^4+h_{113}x_0^4\end{pmatrix}\left(\mathrm{e}^{-x_0}-1\right)m^2\pi^3}{x_0^4\left(x_0^2+16m^2\pi^2\right)}$$
$$+\frac{4K_{16}\sin(\gamma)\left(-h_{112}-h_{114}+h_{113}+h_{111}\right)\left(\mathrm{e}^{-x_0}-1\right)m^2\pi^3}{x_0^2+4m^2\pi^2}+$$
$$-\frac{8\sin(\gamma)\,K_4\left(8h_{113}m^2\pi^2-20h_{114}m^2\pi^2-h_{114}x_0^2-2h_{111}x_0^2+h_{112}x_0^2\right)\left(\mathrm{e}^{-x_0}-1\right)m^3\pi^4}{x_0^3\left(x_0^2+16m^2\pi^2\right)}$$
$$-\frac{16K_{\mathrm{winkler}}S_1^2\pi^5m^4\left(\mathrm{e}^{3x_0}-1\right)\sin(\gamma)}{\left(9x_0^2+16m^2\pi^2\right)\left(9x_0^2+4m^2\pi^2\right)}$$
$$-\frac{8\pi^5m^4\left(8m^2\pi^2-x_0^2\right)\left(\mathrm{e}^{x_0}-1\right)\sin(\gamma)\,K_{\mathrm{pasternak}}}{x_0^2S_1\left(x_0^2+16m^2\pi^2\right)\left(x_0^2+4m^2\pi^2\right)}$$
$$-\frac{\pi^2m\left(8m^3\pi^3h_{114}K_{53}-2m\pi\,K_{53}h_{112}x_0^2-4m^2\pi^2h_{113}K_{63}x_0+K_{63}h_{111}x_0^3\right)\sin(\gamma)}{x_0^3}$$
$$+\frac{8\pi^5\mathrm{e}^{-x_0}m^4\begin{pmatrix}32m^4\pi^4h_{124}+13h_{122}x_0^4-8m^2\pi^2h_{122}x_0^2+18h_{123}x_0^4\\-36m^2\pi^2h_{123}x_0^2+24h_{124}x_0^4-100m^2\pi^2h_{124}x_0^2+9h_{121}x_0^4\end{pmatrix}\left(-1+\mathrm{e}^{x_0}\right)\sin(\gamma)}{\left(x_0^2+16m^2\pi^2\right)x_0^4\left(x_0^2+4m^2\pi^2\right)};$$

$$l_{191}=\frac{2\pi^2m\left(\mathrm{e}^{3x_0}-1\right)\sin(\gamma)\,h_{13}S_1^2}{3\left(4\pi^2m^2+9x_0^2\right)}\left(\frac{\pi m-\pi\,mB_2^2+12x_0+3\pi\,mB_1^2+3x_0B_1}{3}\right);$$

$$l_{192}=\pi x_0\sin(\gamma)\,\cot(\gamma)\,\frac{\left(3S_1^2\left(B_1K_2+K_1\right)+B_1^2K_1-2B_1K_2-3K_1\right)}{2S_1^2}$$
$$+\frac{-2\pi^3m^2\left(\mathrm{e}^{3x_0}-1\right)\sin(\gamma)\,K_{\mathrm{winkler}}S_1^2}{3\left(4\pi^2m^2+9x_0^2\right)}$$
$$+\frac{\pi^2m\left(\mathrm{e}^{x_0}-1\right)\sin(\gamma)\cot(\gamma)\left[x_0\left(3B_1K_{102}S_1^2-B_1^2K_{92}+2K_{92}\right)+2\pi\,m\left(3B_1K_{92}S_1^2+B_1^2K_{102}-2K_{102}\right)\right]}{S_1^2\left(4\pi^2m^2+x_0^2\right)}$$
$$+\frac{x_0\pi\,\sin(\gamma)}{2}\left[\left(B_1^2B_2^2h_{117}+B_2^4h_{118}+B_1^4h_{114}-B_1^2h_{112}-B_2^2h_{115}\right)K_{102}\right.$$
$$\left.+\left(B_1^3h_{113}+B_1B_2^2h_{116}-B_1h_{111}\right)K_{92}\right]$$
$$-\frac{2\pi^2m\left(\mathrm{e}^{x_0}-1\right)\sin(\gamma)\left(x_0B_1K_{\mathrm{pasternak}}+\pi\,m\left(B_1^2K_{\mathrm{pasternak}}+B_2^2K_{\mathrm{pasternak}}-K_{\mathrm{pasternak}}\right)\right)}{S_1\left(4\pi^2m^2+x_0^2\right)}$$
$$+\frac{\mathrm{e}^{-x_0}\pi^2m\left(\mathrm{e}^{x_0}-1\right)\sin(\gamma)\,x_0\left(f_{11}e_{31}K_1+f_{11}e_{32}K_2\right)\;+2\pi\,m\left(f_{11}e_{32}K_1+f_{11}e_{31}K_2\right)}{4\pi^2m^2+x_0^2}$$

$$+\frac{e^{-x_0}\pi^2 m\,(e^{x_0}-1)\sin(\gamma)}{\left(4\pi^2m^2+x_0^2\right)}\,(x_0 f_{11}e_{33}+2\pi m f_{11}e_{34});$$

$$f_{11}e_{31} = \left(B_1h_{111}-2B_1h_{112}+3B_1h_{113}-B_1^3h_{113}+4B_1^3h_{114}-B_1B_2^2h_{116}+2B_1B_2^2h_{117}-4B_1h_{114}\right);$$

$$f_{11}e_{32} = \begin{pmatrix} -B_1^2h_{112}-6B_1^2h_{114}-B_2^2h_{115}+B_2^2h_{116}+3B_1^2h_{113}+B_1^4h_{114} \\ +B_1^2B_2^2h_{117}+B_2^4h_{118}-B_2^2h_{117}-h_{111}+h_{112}-h_{113}+h_{114} \end{pmatrix};$$

$$f_{11}e_{33} = \left(-B_1^3h_{123}-4B_1^3h_{124}-B_1B_2^2h_{126}-2B_1B_2^2h_{127}+B_1h_{121}+2B_1h_{122}+3B_1h_{123}+4B_1h_{124}\right);$$

$$f_{11}e_{34} = \begin{pmatrix} B_1^4h_{124}+B_1^2B_2^2h_{127}+B_2^4h_{128}-B_1^2h_{122}-3B_1^2h_{123}-6B_1^2h_{124} \\ -B_2^2h_{125}-B_2^2h_{126}-B_2^2h_{127}+h_{121}+h_{122}+h_{123}+h_{124} \end{pmatrix};$$

$$l_{20} = \frac{m^2\pi^3\sin(\gamma)}{\left(4\pi^2m^2+9x_0^2\right)}\begin{bmatrix} \frac{4}{3}\left(e^{3x_0}-1\right)\cot(\gamma)\left(3S_1^2+1\right)+4\left(e^{x_0}-1\right)\cot(\gamma) \\ -\frac{S_1^2}{4}\left(e^{2x_0}-1\right)(h_{111}+2h_{112}+4h_{113}+8h_{114}) \end{bmatrix};$$

$$l_{21} = -\frac{2\pi^3\sin(\gamma)\,m^2\left(S_1^3-S_2^3\right)}{3S_1^3}\left(4m^2\pi^2+9\left(\ln\left(\frac{S_2}{S_1}\right)\right)^2\right)^{-1};$$

$$l_{22} = -16\frac{\pi^5\sin(\gamma)\,m^4\left(S_1^3-S_2^3\right)}{S_1^3}\left(16\pi^2m^2+9\left(\ln\left(\frac{S_2}{S_1}\right)\right)^2\right)^{-1}\left(4\pi^2m^2+9\left(\ln\left(\frac{S_2}{S_1}\right)\right)^2\right)^{-1};$$

$$l_{24} = \left(\left(\frac{8\pi^3m^2+2\pi x_0^2}{8\pi^2m^2+2x_0^2}-2\frac{\pi x_0^2}{8\pi^2m^2+2x_0^2}\right)e^{x_0}-\frac{8\pi^3m^2}{8\pi^2m^2+2x_0^2}\right)\sin(\gamma)+\frac{4e^{x_0}\pi^2 m x_0\cos(\gamma)}{8\pi^2m^2+2x_0^2}.$$

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