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ORIGINAL ARTICLE

Nonlinear response of a shear deformable S-FGM shallow spherical shell with ceramic-metal-ceramic layers resting on an elastic foundation in a thermal environment

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ABSTRACT
This article presents an analytical approach to investigate the nonlinear stability of thick, functionally graded material (FGM) shallow spherical shells resting on elastic foundations, subjected to uniform external pressure and exposed to thermal environments. Material properties are assumed to be temperature dependent and graded in the thickness direction according to a Sigmoid power law distribution (S-FGM) in terms of the volume fractions of constituents. Using the first-order shear deformation theory and the Galerkin method, the effects of materials, geometry, elastic foundation parameters, and temperature on the nonlinear response of the thick S-FGM shells are analyzed and discussed in detail.

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Nonlinear response; thick S-FGM shallow spherical shell with ceramic-metal-ceramic layers; elastic foundation; temperature effects

1. Introduction

Functionally graded materials (FGMs) are novel composites usually composed of ceramic and metal constituents. Due to smooth and gradual variation of material constituents, FGMs are capable of reducing or eliminating disadvantageous problems of conventional composites, such as de-bonding and huge stress concentration. By virtue of high stiffness and performance temperature resistance capacity, FGMs can withstand severe thermal environments and are suitable for applications in temperature shielding components, such as aircraft, missile, and aerospace structures. Therefore, stability and vibration of FGM plates and shells in general, and FGM shallow spherical (SS) shells in particular have received much attention.

In addition, spherical shells are nowadays important components widely used as major load-carrying portions in the structures of aircraft, missile, and aerospace vehicles. They also find many applications in various industries, such as shipbuilding, underground structures, and building constructions. Since these shell structures are frequently exposed to severe mechanical and thermal loading conditions, their static and dynamic response are important problems and have received considerable attention.

Eslami and his co-workers [1, 2] made use of approximate analytical solutions, the classical shell theory, and adjacent equilibrium criterion. Brush and Almroth [3] studied the problem, which is performed on the linear buckling of simply supported thin FGM shallow spherical shells (FGM SS shells) and deep spherical shells without elastic foundations and subjected to external pressure and thermal loadings. Recently, Boroujerdy and Eslami [4] studied thermal and thermo-mechanical stability of simply supported thin piezo-FGM SS shells. Bich and Tung [5] presented an investigation on the nonlinear axisymmetrical static response of clamped thin FGM SS shells subjected to external pressure and thermal loads by making use of the classical shell theory and analytical solutions. Also, nonlinear unsymmetrical static and dynamic buckling behavior of thin FGM SS shells have been analyzed by Bich et al. [6] based on an analytical approach and approximate solutions.

In exceptional cases, when the rise of a shell is almost zero, the SS shells are called circular plates. Based on the classical plate theory and shooting method, Ma and Wang [7] and Li et al. [8] dealt with the axisymmetric large deflection bending, thermal and thermo-mechanical post-buckling behavior of thin FGM circular plates without and with initial imperfection. Najafizadeh and Heydari [9] employed a variational method and adjacent equilibrium criterion based on the higher-order shear deformation theory to investigate the linear buckling of clamped FGM circular plates under thermal loads and axisymmetric deformation. Alternatively, thermal buckling of clamped perfect FGM circular plates have been analyzed by Tran et al. [10] utilizing an iso-geometric finite element formulation.

Recently, from the open literature, the nonlinear axisymmetric response of thin FGM SS shells on elastic foundations under uniform external pressure and temperature has been addressed by Duc et al. [11] and the nonlinear thermo-mechanical response of thick axisymmetric shear deformable P-FGM SS shells has been addressed by Tung [12]. As can be seen, the problems of FGM SS shells were still open and should be of interest, e.g., the static and dynamic problems of S-FGM SS shells.

This article presents an analytical investigation on the nonlinear response of shear deformable S-FGM thick SS shell with ceramic-metal-ceramic layers resting on an elastic foundation in a thermal environment. Material properties are assumed to be temperature dependent and graded in the thickness direction according to a Sigmoid power law distribution. Approximate

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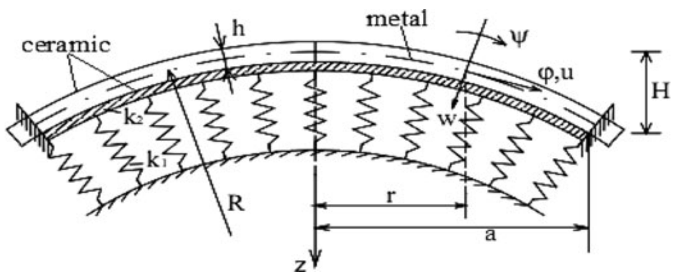

Figure 1. Configuration and the coordinate system of an S-FGM shallow spherical shell (S-FGM SS shell).

solutions are assumed to satisfy immovable clamped boundary conditions and Galerkin method is applied to derive load-deflection relations. The effects of materials, geometrical and foundation stiffness parameters, and temperature dependence of material properties on the nonlinear thermo-mechanical stability of S-FGM SS shells are analyzed and discussed in detail.

2. Governing equations

2.1. *Material components*

Consider a functionally graded shallow spherical shell with radius of curvature R, base radius a, thickness h, and rise of shell H. The shell structure is bounded by the outer of the ceramic-rich surface and the middle of the metal-rich surface with Sigmoid power law distribution (S-FGM) in terms of the volume fractions of constituents. The shell is immovably clamped at the boundary edge and is resting on a Pasternak elastic foundation as shown in Figure 1.

The S-FGM shell is defined in a coordinate system (φ, θ, z) whose origin is located, φ, θ, and z is perpendicular to the middle surface and points outward $-h/2 \le z \le h/2$.

Supposing that the material composition of the shell varies smoothly along the thickness, applying a Sigmoid power law distribution for the shell, the volume fractions of metal and ceramic, V_m and V_c, are assumed as [11, 13]:

$$V_m(z) = \begin{cases} \left(\dfrac{2z+h}{h}\right)^n, -h/2 \le z \le 0 \\ \left(\dfrac{-2z+h}{h}\right)^n, 0 \le z \le h/2 \end{cases}, \quad V_c(z) = 1 - V_m(z), \tag{1}$$

where the volume fraction index n is a non-negative number that defines the material distribution and can be chosen to optimize the structural response. V_m and V_c are volume fractions of metal and ceramic constituents, respectively.

It is assumed that the effective properties P_{eff} of the functionally graded shell, such as the modulus of elasticity E and the coefficient of thermal expansion α, vary in the thickness direction z and can be determined by the linear rule of mixture as:

$$P_{eff} = \mathrm{Pr}_m V_m(z) + \mathrm{Pr}_c V_c(z), \tag{2}$$

where Pr denotes the material properties and subscripts m and c stand for metal and ceramic constituents, respectively. In this study, material properties are assumed to be dependent on temperature.

Specific expressions of material effective properties are obtained by substituting Eq. (1) into Eq. (2) as follows:

$$(E(z,T), \alpha(z,T)) = (E_c(T), \alpha_c(T)) + (E_{mc}(T), \alpha_{mc}(T)) \times \begin{cases} \left(\dfrac{2z+h}{h}\right)^n, -h/2 \le z \le 0 \\ \left(\dfrac{-2z+h}{h}\right), h/2 \le z \le 0, \end{cases} \tag{3}$$

where $E_{mc}(T) = E_m(T) - E_c(T), \alpha_{mc}(T) = \alpha_m(T) - \alpha_c(T)$, and Poisson ratio is assumed to be constant. As can be seen from Eq. (3), at $z = h/2$ and $-h/2$, the surfaces are fully ceramic and, at $z = 0$, the surface is purely metallic. Material properties corresponding to the isotropic shell with $n = 0$ and ceramic component will be increased as n increases as well. A material property Pr, such as elasticity modulus E and thermal expansion coefficient α, can be expressed as a nonlinear function of temperature [14]:

$$\mathrm{Pr}(z,T) = P_0\left(P_{-1}T^{-1} + 1 + P_1T^1 + P_2T^2 + P_3T^3\right), \tag{4}$$

in which $T = T_0 + \Delta T(z)$, ΔT is temperature rise from stress-free initial state, and more generally, $\Delta T = \Delta T(z)$ and $T_0 = 300K$ (room temperature); P_0, P_{-1}, P_1, P_2, and P_3 are coefficients characterizing of the constituent materials.

2.2. *Model of the Pasternak elastic foundation*

The S-FGM SS shell is resting on the elastic foundations. For the elastic foundation, one assumes the two-parameter elastic foundation model proposed by Pasternak. The foundation medium is assumed to be linear, homogeneous, and isotropic. The bonding between the shallow spherical shell and the foundation is perfect and frictionless. If the effects of damping and inertia force in the foundation are neglected, the foundation interface pressure may be expressed as [11, 12]:

$$q_f = k_1 w - k_2 \Delta w, \quad \Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \tag{5}$$

where w is the deflection of the shell, k_1 is Winkler foundation modulus, and k_2 is the shear layer foundation stiffness of the Pasternak model.

2.3. *Theoretical formulation*

The present study uses the first-order shear deformation theory and Galerkin method to obtain governing equations, determine buckling load expression, and post-buckling path of the S-FGM shell exposed to thermal environment and under uniformly distributed external load.

The shell is assumed to be under axisymmetric deformation and the displacement components $\overline{u}, \overline{v}, \overline{w}$ at a distance z from the middle surface are expressed as [11]:

$$\overline{u}(r,z) = u(r) + z\psi(r), \quad \overline{v}(r,z) = 0, \quad \overline{w}(r,z) = w(r) \tag{6}$$

For a S-FGM SS shell it is convenient to introduce a variable r, referred to as the radius of parallel circle with the base of shell and defined by $r = R\sin\varphi$ and ψ is the rotation of a normal to the middle surface. Due to shallowness of the spherical shell, it is approximately assumed that $\cos\varphi = 1$, $Rd\varphi = dr$, and $R = a^2/(2H)$.

The strain components at a distance z from the middle surface are defined as [11]:

$$\varepsilon_r = \varepsilon_{r0} + z\chi_r, \quad \varepsilon_\theta = \varepsilon_{\theta 0} + z\chi_\theta, \quad \varepsilon_{rz} = \psi + \frac{\partial w}{\partial r}, \tag{7}$$

where

$$\varepsilon_{r0} = \frac{\partial u}{\partial r} - \frac{w}{R} + \frac{1}{2}\frac{\partial^2 w}{\partial r^2}, \quad \varepsilon_{\theta 0} = \frac{u}{r} - \frac{w}{R}, \quad \chi_r = \frac{\partial \psi}{\partial r}, \quad \chi_\theta = \frac{\psi}{r}.$$

Here, ε_{r0}, $\varepsilon_{\theta 0}$ are strains components at the middle surface in the meridional and circumferential directions, respectively; χ_r, χ_θ are curvature components.

Applying Hooke law for S-FGM SS shell, the following is obtained:

$$\begin{aligned}
\sigma_r &= \frac{E(z,T)}{1-\nu^2}\left[\varepsilon_r + \nu\varepsilon_\theta - (1+\nu)\alpha(z,T)\Delta T\right],\\
\sigma_\theta &= \frac{E(z,T)}{1-\nu^2}\left[\varepsilon_\theta + \nu\varepsilon_r - (1+\upsilon)\alpha(z,T)\Delta T\right],\\
\sigma_{rz} &= \frac{E(z,T)}{2(1+\upsilon)}\varepsilon_{rz},
\end{aligned} \tag{8}$$

Where ΔT denotes temperature difference between initial and final states.

The force and moment resultants of the shell are expressed in terms of the stress components through the thickness as:

$$\begin{aligned}
(N_r, N_\theta) &= \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta)\, dz,\\
(M_r, M_\theta) &= \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta)\, z dz, \quad Q_r = K_s \int_{-h/2}^{h/2} \sigma_{rz} dz.
\end{aligned} \tag{9}$$

where K_s is correction coefficient and chosen to be 5/6.

Substituting Eqs. (7) and (8) into Eq. (9) and performing the integrations, we obtain:

$$\begin{aligned}
N_r &= \frac{E_1}{1-\nu^2}(\varepsilon_{r0} + \nu\varepsilon_{\theta 0}) + \frac{E_2}{1-\nu^2}(\chi_r + \nu\chi_\theta) - \frac{\Phi_0}{1-\nu},\\
N_\theta &= \frac{E_1}{1-\nu^2}(\varepsilon_{\theta 0} + \nu\varepsilon_{r0}) + \frac{E_2}{1-\nu^2}(\chi_\theta + \nu\chi_r) - \frac{\Phi_0}{1-\nu},\\
Q_r &= \frac{K_s E_1}{2(1+\nu)}\left(\psi + \frac{\partial w}{\partial r}\right),\\
M_r &= \frac{E_2}{1-\nu^2}(\varepsilon_{r0} + \nu\varepsilon_{\theta 0}) + \frac{E_3}{1-\nu^2}(\chi_r + \nu\chi_\theta) - \frac{\Phi_1}{1-\nu},\\
M_\theta &= \frac{E_2}{1-\nu^2}(\varepsilon_{\theta 0} + \nu\varepsilon_{r0}) + \frac{E_3}{1-\nu^2}(\chi_\theta + \nu\chi_r)\\
&\quad - \frac{\Phi_1}{1-\nu}.,
\end{aligned} \tag{10A}$$

where

$$\begin{aligned}
E_1 &= \left(E_c(T) + \frac{E_{mc}(T)}{n+1}\right)h, \; E_2 = 0,\\
E_3 &= \frac{E_c}{12}h^3 + \frac{E_{mc}}{2(n+1)(n+2)(n+3)}h^3.,\\
\Phi_0, \Phi_1 &= \int_{-h/2}^{0}\left[E_c(T) + E_{mc}(T)\left(\frac{2z+h}{h}\right)^n\right]\\
&\quad\times\left[\alpha_c(T) + \alpha_{mc}(T)\left(\frac{2z+h}{h}\right)^n\right]\Delta T(1,z)\,dz\\
&\quad+\int_0^{h/2}\left[E_c(T) + E_{mc}(T)\left(\frac{-2z+h}{h}\right)^n\right]\\
&\quad\times\left[\alpha_c(T) + \alpha_{mc}(T)\left(\frac{-2z+h}{h}\right)^n\right]\\
&\quad\times\Delta T(1,z)\,dz.
\end{aligned} \tag{10b}$$

The nonlinear equilibrium equations of thick S-FGM SS shell resting on elastic foundations based on the first-order shear deformation theory are [12]:

$$\frac{\partial(rN_r)}{\partial r} - N_\theta = 0, \tag{11a}$$

$$\frac{\partial(rM_r)}{\partial r} - M_\theta - rQ_r = 0, \tag{11b}$$

$$\begin{aligned}
&\frac{\partial(rQ_r)}{\partial r} + \frac{r}{R}(N_r + N_\theta) + \frac{\partial\left(rN_r\frac{\partial w}{\partial r}\right)}{\partial r}\\
&+r\left(q - q_f\right) = 0
\end{aligned} \tag{11c}$$

Setting Eq. (7) into Eq. (10) and then substituting the obtained expressions and Eq. (5) into Eqs. (11a)–(11c) lead to systems of equilibrium equations of the shell as follows:

$$\begin{aligned}
L_1 \equiv &\frac{E_1}{1-\nu^2}\Bigg[ru_{,rr} + u_{,r} - \frac{u}{r} - (1+\upsilon)r\frac{w_{,r}}{R} + \frac{1}{2}(1-\nu)w_{,r}^2\\
&+ r\left(w_{,r}w_{,rr}\right)\Bigg] = 0,
\end{aligned} \tag{12a}$$

$$\begin{aligned}
L_2 \equiv &\frac{E_3}{1-\nu^2}\left(r\psi_{,rr} + \psi_{,r} - \frac{\psi}{r}\right) - \frac{K_s E_1}{2(1+\nu)}\left(r\psi + rw_{,r}\right) = 0,\\
L_3 \equiv &\frac{K_s E_1}{2(1+\upsilon)}\left[\psi + r\psi_{,r} + w_{,r} + rw_{,rr}\right]\\
&+\frac{E_1}{R(1-\upsilon)}\left(ru_{,r} + u - \frac{2}{R}rw + \frac{r}{2}w_{,r}^2\right)
\end{aligned} \tag{12b}$$

$$\begin{aligned}
&+\frac{E_1}{1-\nu^2}\left[\begin{array}{l}(1+\upsilon)u_{,r}w_{,r} - \frac{1+\upsilon}{R}\left(w + rw_{,r}\right)w_{,r} - (1+\upsilon)\frac{r}{R}w\\ \left(w_{,rr}\right) + ru_{,rr}w_{,r} + \left(ru_{,r} + \nu u\right)w_{,rr} + \frac{1}{2}w_{,r}^3\\ +\frac{3}{2}rw_{,r}^2 w_{,rr}\end{array}\right]\\
&+rq - k_1 rw + k_2\left(rw_{,rr} + w_{,r}\right) - \frac{2r\Phi_0}{R(1-\upsilon)}\\
&-\frac{\Phi_0}{1-\upsilon}\left[w_{,r} + rw_{,rr}\right] = 0
\end{aligned} \tag{12c}$$

3. Solution of the problems

The shell is assumed to be clamped and immovable in the meridional direction at the boundary edges and under axisymmetric deformation. Boundary conditions are expressed as [15, 16]:

$$\psi = 0 \quad \text{at } r = 0 \tag{13a}$$

$$w = 0, \quad \psi = 0, \quad u = 0 \text{ at } \quad r = a. \tag{13b}$$

To satisfy the above proposed boundary conditions, the following approximate solutions for displacement components and rotation are assumed [17]:

$$u = U\frac{r(a-r)}{a^2}, \quad \psi = \Psi\frac{r\left(a^2 - r^2\right)}{a^3}, \quad w = W\frac{\left(a^2 - r^2\right)^2}{a^4}. \tag{14}$$

Here, U, Ψ are coefficients which should be determined and W is the amplitude of deflection.

Setting these solutions into the modified equilibrium equations and applying Galerkin procedure, i.e.:

$$\int_0^a L_1' r\,(a-r)\,dr = 0,\qquad \int_0^a L_2' r\left(a^2-r^2\right)dr = 0,$$
$$\int_0^a L_3'\left(a^2-r^2\right)^2 dr = 0. \tag{15}$$

Performing the integrations, the following is obtained:

$$\overline{U} - \frac{88}{105}(1+\upsilon)\lambda_2\overline{W} - \frac{4}{315\lambda_1}(23-41\upsilon)\overline{W^2} = 0, \tag{16a}$$

$$\left[\frac{2}{3\lambda_1^2}\overline{E_3} - (1-\upsilon)\frac{K_s\overline{E_1}}{48}\right]\Psi + (1-\upsilon)\frac{K_s\overline{E_1}\,\overline{W}}{12\lambda_1} = 0, \tag{16b}$$

$$\begin{aligned} q = &-\frac{132}{105}\frac{\overline{E_1}\lambda_2}{(1-\upsilon)\lambda_1^2}\overline{U} - K_s\frac{\overline{E_1}}{2(1+\upsilon)\lambda_1}\Psi \\ &-\frac{12}{315}\frac{\overline{E_1}(23-41\upsilon)}{\lambda_1^3(1-\upsilon^2)}\overline{U}\,\overline{W} + \\ &+\left[2\frac{K_s\overline{E_1}}{(1+\upsilon)\lambda_1^2} + \frac{504}{105}\frac{\overline{E_1}\lambda_2^2}{(1-\upsilon)\lambda_1^2} - 4\frac{\overline{\Phi}_0}{(1-\upsilon)\lambda_1^2}\right. \\ &\left.+\frac{1}{20}\frac{K_1\overline{E_1}}{(1-\upsilon^2)\lambda_1^4} + \frac{K_2\overline{E_1}}{3(1-\upsilon^2)\lambda_1^4}\right]\overline{W} - \\ &-\frac{24}{5}\frac{\overline{E_1}\lambda_2}{(1-\upsilon)\lambda_1^3}\overline{W^2} + \frac{384}{105}\frac{\overline{E_1}}{(1-\upsilon^2)\lambda_1^4}\overline{W^3} + \frac{4\lambda_2\overline{\Phi}_0}{(1-\upsilon)\lambda_1}, \end{aligned} \tag{16c}$$

where

$$\overline{E_1} = E_1/h,\ \overline{E_3} = E_3/h^3,\ \overline{U} = U/h,$$
$$\overline{W} = W/h,\ \overline{\Phi}_0 = \Phi_0/h,\ \lambda_1 = a/h,\ \lambda_2 = H/a,$$
$$K_1 = \frac{12\left(1-\nu^2\right)a^4}{E_1h^2}k_1,\ K_2 = \frac{12\left(1-\nu^2\right)a^2}{E_1h^2}k_2.$$

From Eqs. (16a) and (16b), the two equations of $\overline{U}$ and Ψin terms of $\overline{W}$ can be obtained as:

$$\overline{U} = \frac{88}{105}(1+\nu)\lambda_2\overline{W} + \frac{4}{315}\frac{(23-41\nu)}{\lambda_1}\overline{W^2},$$
$$\Psi = -\frac{4(1-\nu)K_s\overline{E_1}\lambda_1}{32\overline{E_3} - K_s\overline{E_1}\lambda_1^2 + K_s\overline{E_1}\lambda_1^2\nu}\overline{W}. \tag{17}$$

In the study, temperature is assumed to be raised uniformly, i.e., temperature increases from initial state T_0 to final state T_f and the change of temperature $\Delta T = T_f - T_i$ is independent on thickness variable z of the shell.

From Eq. (10b), thermal parameter Φ_0 can be expressed as:

$$\Phi_0 = Ph\Delta T, \tag{18}$$

where

$$P = E_c\alpha_c + \frac{E_c\alpha_{mc} + E_{mc}\alpha_c}{n+1} + \frac{E_{mc}\alpha_{mc}}{2n+1}.$$

Setting Eqs. (17) and (18) into Eq. (16c) leads to:

$$\begin{aligned} q = &\frac{4\lambda_2P\Delta T}{(1-\upsilon)\lambda_1} \\ &+\left[\begin{aligned}&\frac{-3872}{3675}\frac{E_1\overline{\lambda_2^2(1+\upsilon)}}{(1-\nu)\lambda_1^2} \\ &+\frac{2K_s^2\,\overline{E_1^2}(1-\nu)}{\left(32\overline{E_3} - K_sE_1\overline{\lambda_1^2} + K_s\overline{E_1}\lambda_1^2\nu\right)(1+\nu)} + \frac{2K_s\overline{E_1}}{(1+\upsilon)\lambda_1^2} \\ &+\frac{24}{5}\frac{\overline{E_1}\lambda_2^2}{(1-\nu)\lambda_1^2} - 4\frac{P\Delta T}{(1-\nu)\lambda_1^2} + \frac{1}{20}\frac{K_1\overline{E_1}}{\lambda_1^4\left(1-\nu^2\right)} \\ &+\frac{1}{3}\frac{K_2\overline{E_1}}{\left(1-\nu^2\right)\lambda_1^4}\end{aligned}\right]\overline{W} \\ &+\left[-\frac{176}{11025}\frac{\overline{E_1}\lambda_2(23-41\nu)}{(1-\upsilon)\lambda_1^3} - \frac{352}{11025}\frac{\overline{E_1}\lambda_2(23-41\nu)}{(1-\nu)\lambda_1^3}\right. \\ &\left.-\frac{24}{5}\frac{\overline{E_1}\lambda_2}{(1-\upsilon)\lambda_1^3}\right]\overline{W^2} \\ &+\left[\frac{128}{35}\frac{\overline{E_1}}{(1-\nu^2)\lambda_1^4} - \frac{16}{33075}\frac{\overline{E_1}(23-41\nu)^2}{(1-\nu^2)\lambda_1^4}\right]\overline{W^3}. \end{aligned} \tag{19}$$

Respectively to:

$$q = e_1 + e_2\overline{W} + e_3\overline{W^2} + e_4\overline{W^3}, \tag{20}$$

where

$$e_1 = \frac{4\lambda_2P\Delta T}{(1-\upsilon)\lambda_1}\quad e_4 = \left[\frac{128}{35}\frac{\overline{E_1}}{(1-\nu^2)\lambda_1^4} - \frac{16}{33075}\frac{\overline{E_1}(23-41\nu)^2}{(1-\nu^2)\lambda_1^4}\right];$$

$$e_2 = \left[\begin{aligned}&\frac{-3872}{3675}\frac{E_1\overline{\lambda_2^2(1+\upsilon)}}{(1-\nu)\lambda_1^2} + \frac{2K_s^2\,\overline{E_1^2}(1-\nu)}{\left(32\overline{E_3} - K_sE_1\overline{\lambda_1^2} + K_s\overline{E_1}\lambda_1^2\nu\right)(1+\nu)} \\ &+\frac{2K_s\overline{E_1}}{(1+\upsilon)\lambda_1^2} + \frac{24}{5}\frac{\overline{E_1}\lambda_2^2}{(1-\nu)\lambda_1^2} \\ &-4\frac{P\Delta T}{(1-\nu)\lambda_1^2} + \frac{1}{20}\frac{K_1\overline{E_1}}{\lambda_1^4\left(1-\nu^2\right)} + \frac{1}{3}\frac{K_2\overline{E_1}}{\left(1-\nu^2\right)\lambda_1^4}\end{aligned}\right]$$

$$\begin{aligned} e_3 = &\left[-\frac{176}{11025}\frac{\overline{E_1}\lambda_2(23-41\nu)}{(1-\upsilon)\lambda_1^3} - \frac{352}{11025}\frac{\overline{E_1}\lambda_2(23-41\nu)}{(1-\nu)\lambda_1^3}\right. \\ &\left.-\frac{24}{5}\frac{\overline{E_1}\lambda_2}{(1-\upsilon)\lambda_1^3}\right]. \end{aligned}$$

Equation (20) is an explicit expression of load–deflection curves for the clamped immovable shells resting on Pasternak elastic foundations and is subjected to combined pressure and thermal loadings.

As the temperature is maintained at initial value $T = T_0$, i.e., $\Delta T = 0$, pressure-loaded shells may exhibit an extremum type buckling behavior and extremum points of pressure-deflection curves can be determined from the condition:

$$\frac{dq}{d\overline{W}} = e_2 + 2e_3\overline{W} + 3e_4\overline{W^2} = 0 \tag{21}$$

which gives,

$$\overline{W}_{1,2} = \frac{-e_3 \mp \sqrt{e_3^2 - 3e_2e_4}}{3e_4}, \tag{22}$$

and critical buckling pressures of the shells are obtained as:

$$q_{cr} = q\left(\overline{W_1}\right) = e_1 + e_2\overline{W_1} + e_3\overline{W_1}^2 + e_4\overline{W_1}^3, \tag{23}$$

providing material and shell geometry parameters to satisfy the condition:

$$e_3^2 - 3e_2e_4 \geq 0. \tag{24}$$

Conversely, in the case of $\Delta T \neq 0$ and due to temperature dependence of material properties, the effects of elevated temperature are included in all terms at the right-hand side of Eq. (20), and thermo-mechanically loading of the shells may experience a bifurcation-type buckling behavior and corresponding critical buckling pressures are predicted as:

$$q_{cr}^{\Delta T} = e_1.$$

Subsequently, specialization of Eq. (20) for the case of clamped immovable FGM circular plates resting on elastic foundations and exposed to thermal environments, i.e., $q = 0$ and $[\lambda_2 = H/a = 0,]$ gives the following relation:

$$\Delta T = \frac{(1-\nu)\,\lambda_1^2}{4P}\left(e_2' + e_3'\,\overline{W} + e_4'\,\overline{W^2}\right), \tag{25}$$

where e_2', e_3', e_4' are received by specialization of e_2, e_3, e_4, respectively, in which λ_2 is eliminated and $\overline{\Phi_0}$ is eliminated in e_2.

Then, ΔT_{cr} for the circular plates can be obtained by setting $\overline{W} \to 0$ from (25):

$$\Delta T_{cr} = \frac{(1-\nu)\,\lambda_1^2}{4P}\left[\begin{array}{l} \frac{2(1-\nu)K_s^2\,\overline{E_1}}{\left(32\,\overline{E_3} - K_s\,\overline{E_1}\,\lambda_1^2 + K_s\,\overline{E_1}\,\lambda_1^2\nu\right)(1+\nu)} \\ +2\frac{K_s\,\overline{E_1}}{(1+\nu)\lambda_1^2} \\ +\frac{1}{20}\frac{K_1\,\overline{E_1}}{(1-\nu^2)\lambda_1^4} + \frac{1}{3}\frac{\overline{E_1}\,K_2}{(1-\nu^2)\lambda_1^4} \end{array}\right]. \tag{26}$$

In the case of temperature independent material properties, Eqs. (25) and (26) are closed-form expressions of thermal buckling loads and post-buckling curves of the FGM circular plates, respectively. In contrast, as properties of constituent materials in FGM are temperature dependent, Eqs. (25) and (26) are implicit expressions of temperature-deflection relation and critical buckling temperature change, respectively, and an iterative process is adopted to obtain critical buckling temperature and the post-buckling equilibrium curves of thermally loaded FGM circular plates.

4. Numerical results

4.1. *Comparison study*

To validate the proposed approach, thermal buckling of a clamped FGM circular plate under uniform temperature rise and without elastic foundations is considered. The critical buckling temperature changes are calculated by closed-form relation (26) and compared in Figure 2 with results obtained [10] by Tran et al. based on an iso-geometric finite element approach within the framework of the higher-order shear deformation plate theory. Further, n^* is the volume fraction index in case of V_m and V_c are interchanged in Eq. (1), i.e., $V_c(z) = ((2z+h)/(2h))^{n^*}$. The combination of materials consist of aluminum (Al) and alumina (Al_2O_3). Temperature independent elasticity modulus and thermal expansion coefficient are $E_m = 70$ GPa, $\alpha_m = 23 \times 10^{-6}\ {}^0C^{-1}$ for aluminum and $E_c = 380$ GPa, $\alpha_c = 7.4 \times 10^{-6}\ {}^0C^{-1}$ for alumina, whereas Poisson's ratio is a constant $\nu = 0.3$ for both materials. It is evident that an excellent agreement is obtained in this comparison.

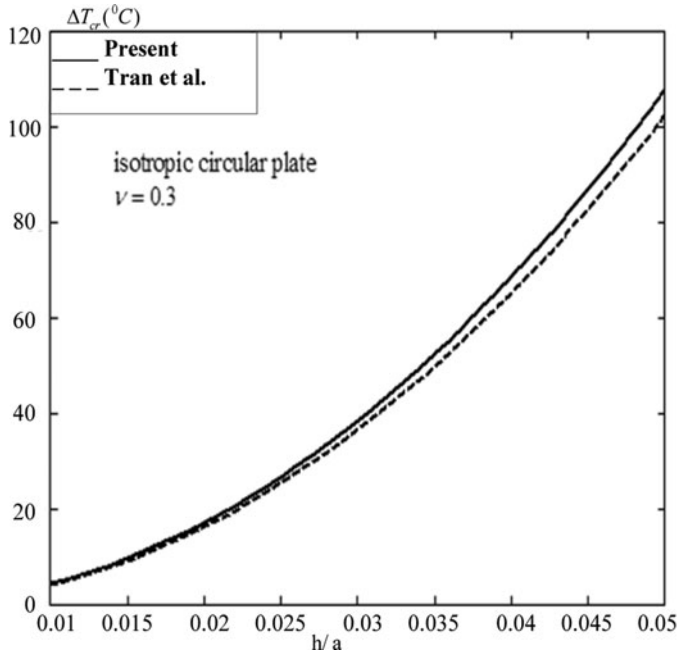

Figure 2. Comparison of the critical buckling temperature change for clamped isotropic FGM circular plate under uniform temperature rise.

As a second example for verification, the nonlinear response of the S-FGM SS shells under uniform external pressure with and without elastic foundations is considered and compared in Figure 3, with results obtained by Tung [12] for the P-FGM SS shells.

The FGM SS shells are composed of silicon nitride (Si_3N_4) and stainless steel (SUS304). Specific values of these coefficients of silicon nitride and stainless steel quoted in Table 1 are given by Reddy and Chin [17] and are calculated as a nonlinear function of temperature by using Eq. (4); the Poisson's ratio is assumed to

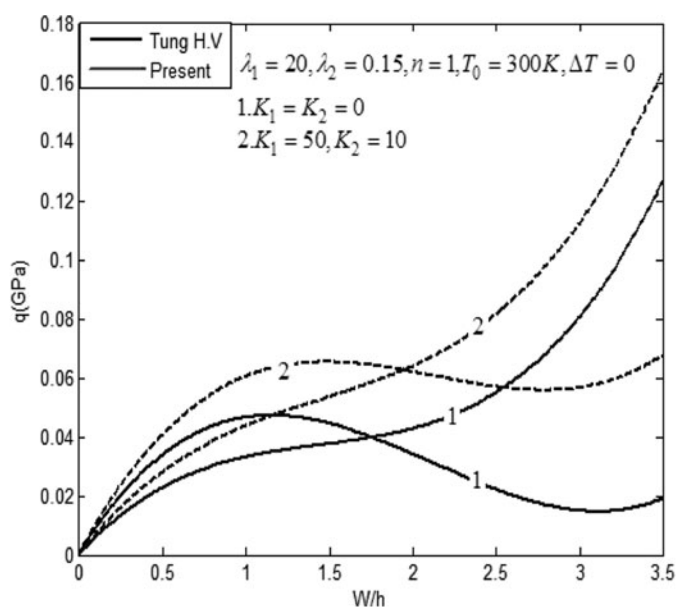

Figure 3. Comparison of the nonlinear response change for S-FGM SS shell and P-FGM SS shell.

Table 1. Material properties of the constituent materials of the considered SS-FGM shell.

Material	Property	P_0	P_{-1}	P_1	P_2	P_3
Si_3N_4 (ceramic)	E (Pa)	348.43e9	0	−3.70e-4	2.160e-7	−8.946e-11
	ρ (kg/m^3)	2370	0	0	0	0
	α (K^{-1})	5.8723e-6	0	9.095e-4	0	0
	k (W/mK)	13.723	0	0	0	0
	ν	0.24	0	0	0	0
SUS304 (metal)	E (Pa)	201.04e9	0	3.079e-4	−6.534e-7	0
	ρ (kg/m^3)	8166	0	0	0	0
	α (K^{-1})	12.330e-6	0	8.086e-4	0	0
	k (W/mK)	15.379	0	0	0	0
	ν	0.3177	0	0	0	0

be a constant $\nu = 0.3$. The shells are assumed to be clamped and immovable along the boundary edge.

As can be seen, Figure 3 shows that, initially at the buckling period, the loading capability of FGM SS shell is better than the S-FGM SS shell, but at the post-buckling period this is the complete opposite, when they have the same shell thickness. This means that the loading capability of S-FGM SS shell at the post-buckling period is better than the loading capability of P-FGM SS shell.

4.2. *The nonlinear response of axisymmetrically deformed S-FGM SS shells*

The remainder of this section presents numerical results for S-FGM SS shells composed of silicon nitride (Si_3N_4) and stainless steel (SUS304), such as above. The temperature-independent material properties will be calculated at room temperature $T_0 = 300K$.

In what follows, the nonlinear response of axisymmetrically deformed S-FGM SS shells will be analyzed. Unless otherwise specified, the thermal environment is maintained at reference value T_0, i.e., $\Delta T = 0$, and the shell is free from elastic foundation interaction, i.e., $K_1 = K_2 = 0$. In characterizing the behavior of the shells, deformations in which the central region of a shell moves toward the plane that contains the periphery of the shell are referred to as inward deflections (positive deflections). Deformations in the opposite direction are referred to as outward deflections (negative deflections).

The effects of foundation stiffness on the critical buckling loads and post-buckling load-deflection curves for S-FGM SS shells subjected to uniform external pressure are graphically illustrated in Figure 4. It indicates that the critical buckling pressures are enhanced due to an increase of stiffness parameters of elastic foundations, especially nondimensional stiffness of the shear layer of the Pasternak foundation model.

Figure 5 shows the effects of the volume fraction index n and elastic foundation on the nonlinear response of S-FGMSS shells under uniform external pressure. As can be seen, ceramic-rich shells have both higher critical loads and more severe snap-through instability. The pressure-deflection curves become higher and more stable, i.e., very benign snap-through phenomenon, as the FGM SS shell is supported by an elastic foundation.

The effects of the shell rise to base radius ratio H/a on the nonlinear stability of S-FGM SS shells are analyzed in Figure 6.

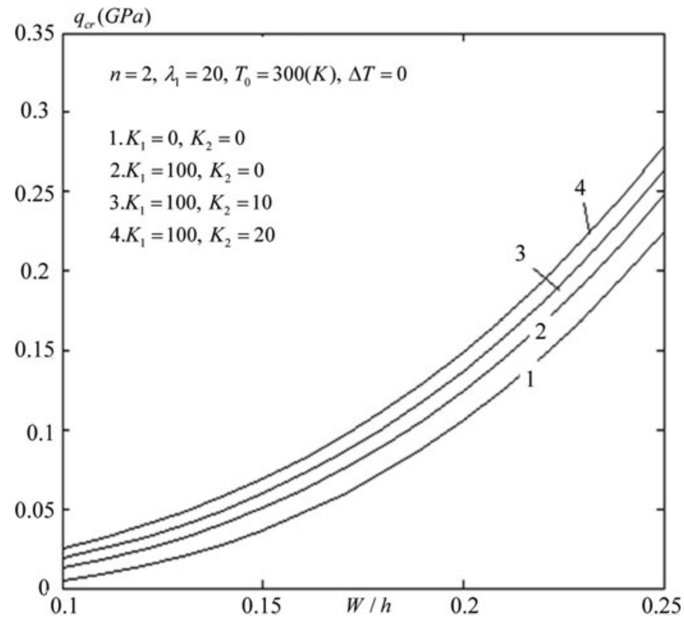

Figure 4. Effects of stiffness parameters of elastic foundations on the critical buckling pressure of the S-FGM SS shell.

It is clear that the nonlinear response of shells is very sensitive to variation of H/a ratio. Specifically, an increase in H/a ratio gives higher buckling loads followed by a more intense snap-through phenomenon.

Figure 7 plotted as counter parts of Figure 6 for the case of $K_1 = 50$, $K_2 = 20$, and Figure 8 depicted with various values of nondimensional stiffness parameters K_1, K_2 show pronounced effects of the support of elastic foundations on the nonlinear response of pressure-loaded S-FGM SS shells. As can be observed, pressure-deflection curves are higher and more stable in the presence of elastic foundations. In addition, parameter K_2 of the Pasternak foundation model has two more sensitive effects on the loading carrying capacity of FGM SS shells.

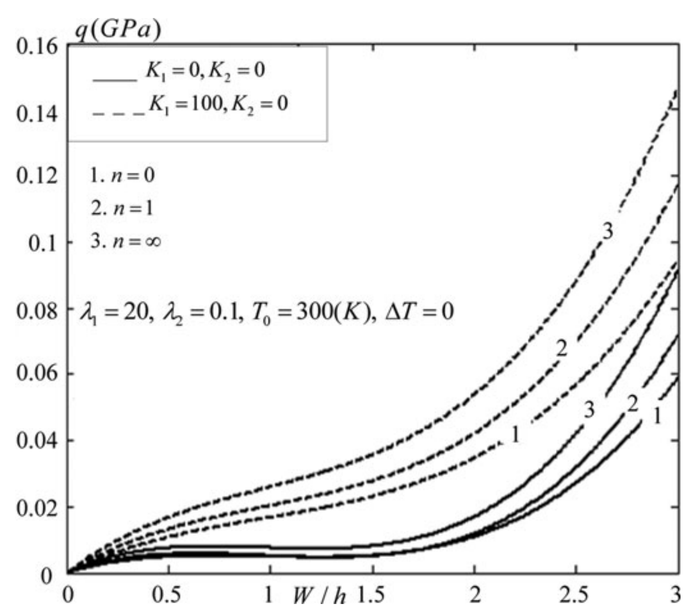

Figure 5. The effects of the volume fraction index n and elastic foundation on the nonlinear response of S-FGM SS shells under uniform external pressure.

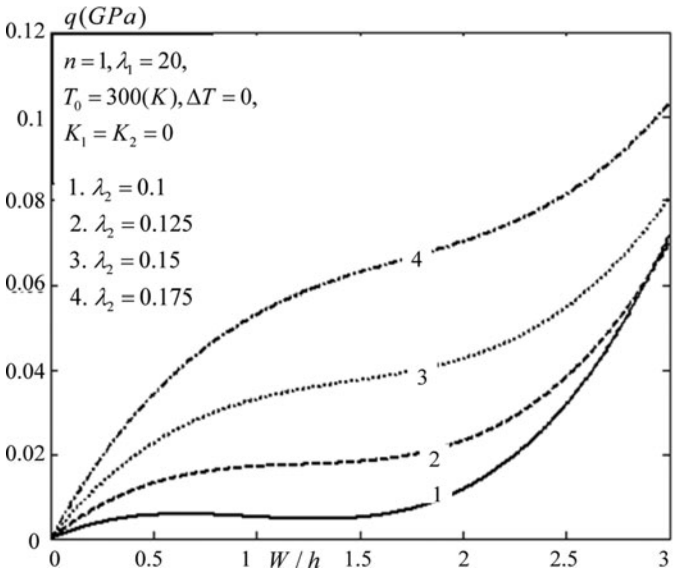

Figure 6. The effects of radius ratio H/a on the nonlinear stability of S-FGM SS shells under uniform external pressure without an elastic foundation.

The effects of thermal environments and temperature dependence of material properties on the nonlinear thermo-mechanical response of S-FGM SS shells are illustrated in Figures 9 and 10. Due to the presence of a thermal environment, pressure-loaded shells exhibit a bifurcation-type buckling behavior where bifurcation point pressure is increased as temperature change becomes higher. It can be explained that pre-existent thermal loading makes the shell surface deflect outwards (negative deflection) and curvature of the shell developed prior to application of external pressure. As temperature dependence of material properties is incorporated, the shells have higher buckling pressures and lower loading capacity in the deep region of post-buckling behavior. Figure 10, plotted as counterparts of Figure 9 for the case of $K_1 = 100$, $K_2 = 10$, again indicates very useful effects of elastic foundations on the stability of S-FGM SS shells subjected to combined thermo-mechanical loads.

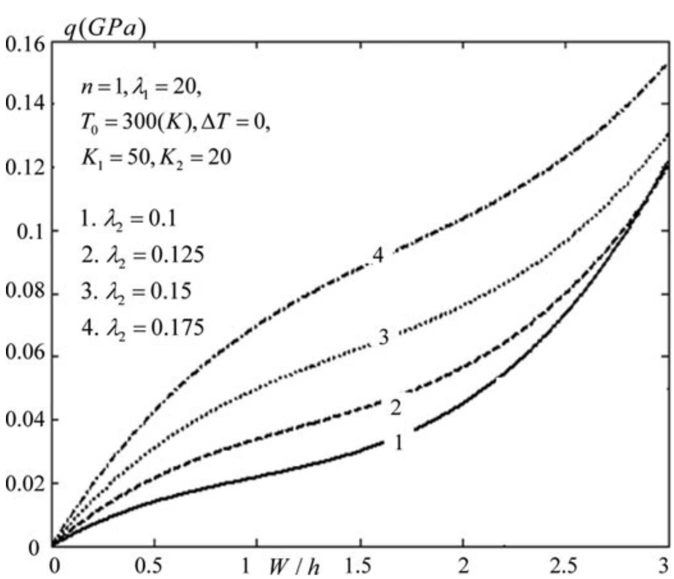

Figure 7. The effects of radius ratio H/a on the nonlinear stability of S-FGM SS shells resting on an elastic foundation under uniform external pressure.

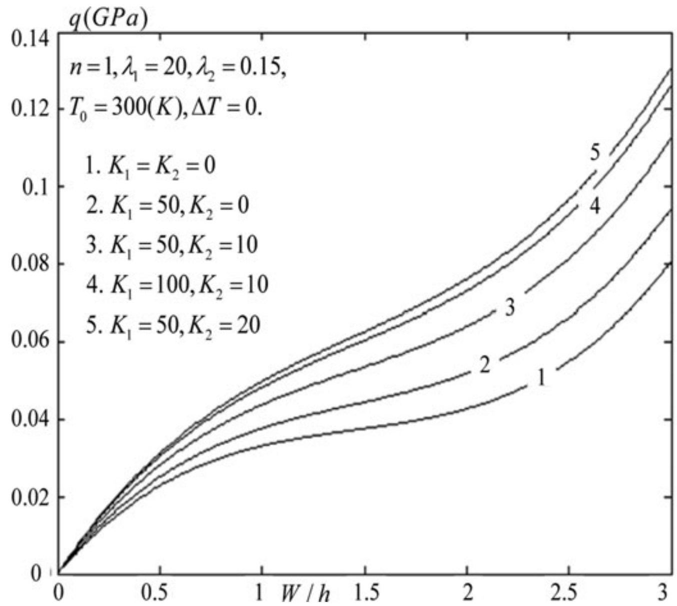

Figure 8. Effects of nondimensional stiffness parameters K_1, K_2 on the nonlinear response of S-FGM SS shells.

4.3. *FGM circular plates*

In exceptional cases, when the rise of a shell is almost zero, $\lambda_2 = 0$, Figure 11, plotted by using Eq. (23) for the case of $\Delta T = 0$ and $[H/a = 0]$, shows pressure deflection curves versus various values of n index and foundation stiffness (K_1, K_2) for S-FGM circular plates under uniform external pressure. It is obvious that equilibrium paths are stable with no snap-through phenomenon, and loading capacity of S-FGM circular plates is remarkably improved due to the support of elastic foundations.

Finally, the effects of elastic foundations on the thermal post-buckling of geometrically S-FGM circular plates are considered in Figure 12. As expected, the figure indicates that deteriorative influences of temperature-dependent material properties on

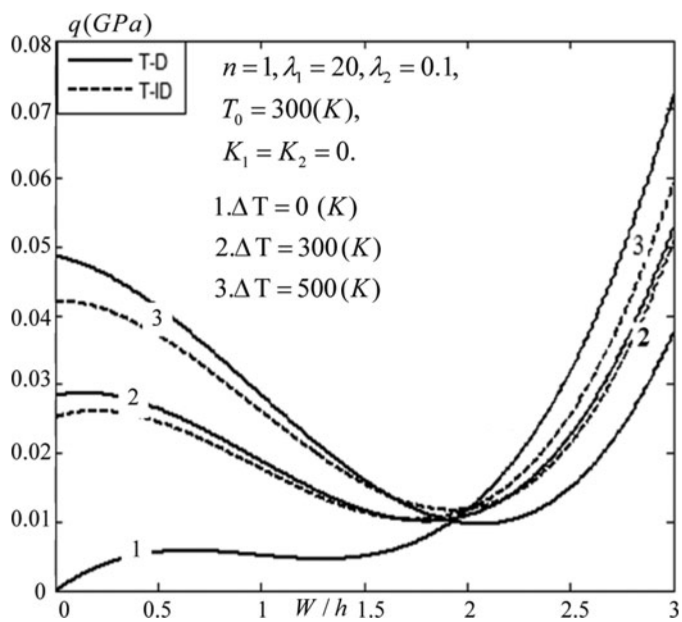

Figure 9. Effects of thermal environments on the nonlinear thermo-mechanical response of S-FGM SS shells.

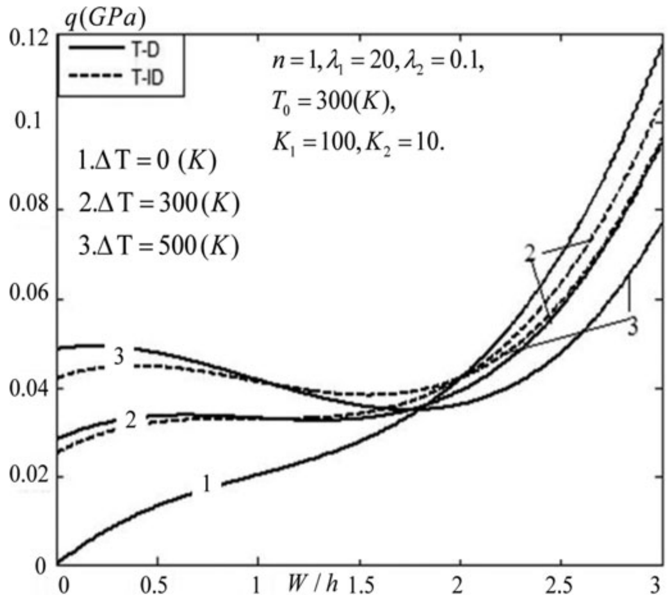

Figure 10. Counterparts of Figure 9 for the case of $K_1 = 100$, $K_2 = 10$.

reduction of thermal loading carrying capability of S-FGM circular plates become more pronounced as S-FGM plates are supported by foundations with higher values of stiffness parameters.

4.4. S-FGM spherical shells with ceramic-metal-ceramic ($c - m - c$) layers and metal-ceramic-metal ($m - c - m$) layers

The study can completely expand for the case of inner and outer surface metallic and mid-surface ceramic. Figures 13 and 14 show the effects of nondimensional stiffness parameters K_1, K_2 and the volume fraction index n on the critical bucking pressures of the shells in the same shape, dimensions, impact force, and thermal environment. As can be seen, Figure 13 shows the influence of the K_1, K_2 on the critical buckling pressures for the distribution of grades in the spherical shell with no considerable difference. From Figure 14, we can conclude that when the

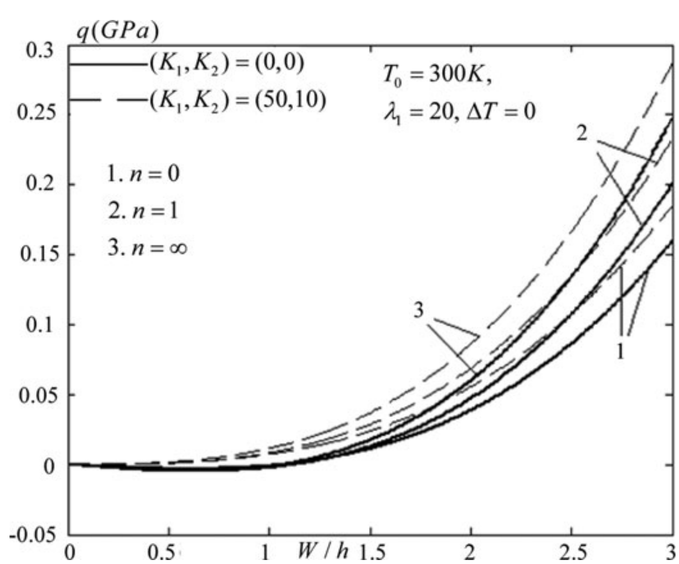

Figure 11. Effects of material distribution and foundation stiffness on the nonlinear response of S-FGM circular plates under uniform external pressure.

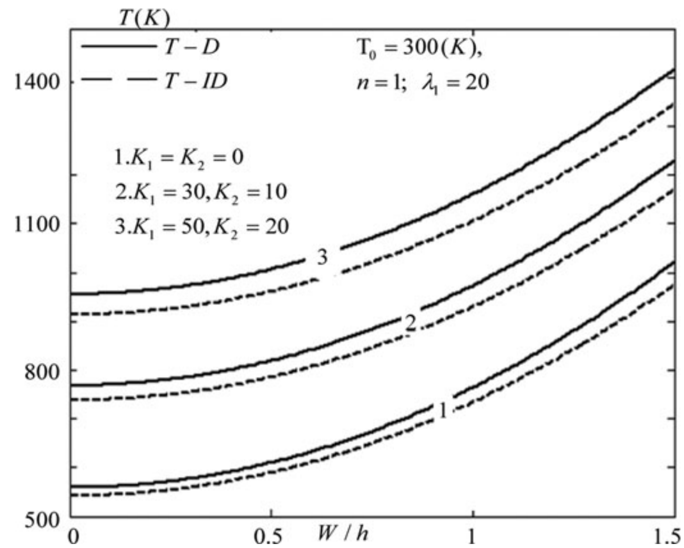

Figure 12. Effects of elastic foundations and temperature dependent properties on the thermal post-buckling of the S-FGM circular plates.

higher increasing of the volume fraction index n $(n > 1)$, corresponding to the higher load capacity of the shells, concurrently the buckling load capacities of the shell with $c - m - c$ layers are better than the one with $m - c - m$ layers, especially in the period of post-buckling. This occurs due to the elasticity modulus E of ceramic higher than metal.

4.4. S-FGM SS with two different boundary conditions

We consider an additional case if the shell is assumed to be clamped and freely movable in the meridional direction at the boundary edges and under axisymmetric deformation. Figure 15 shows the effects of in-plane restraint conditions on the nonlinear response of S-FGM SS shell with two different boundary conditions. In comparison with the FM case, the shells with immovable clamped edge (IM) on elastic foundations have a comparatively higher capability of carrying external pressure

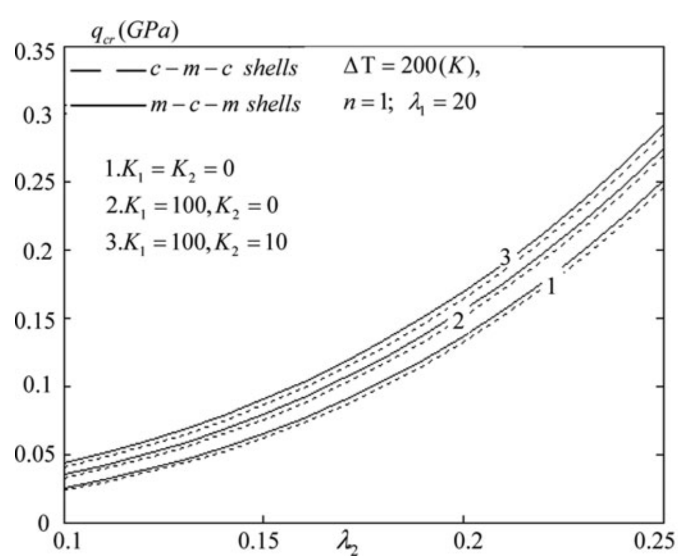

Figure 13. The effects of nondimensional stiffness parameters K_1, K_2 on the critical bucking pressures of the shells.

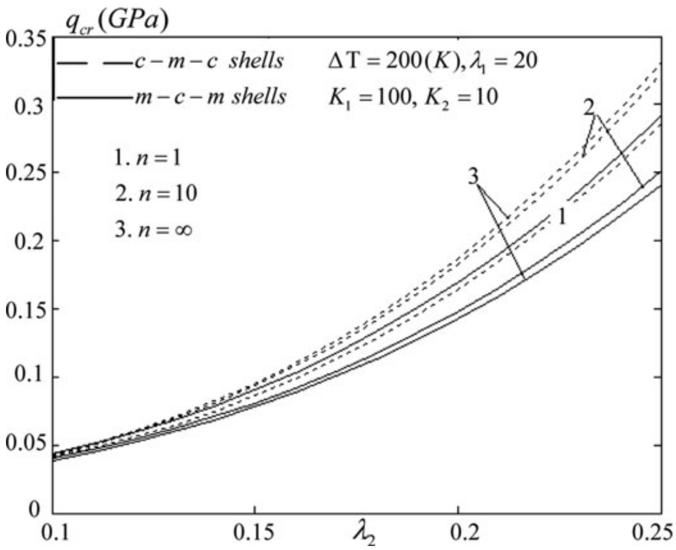

Figure 14. The effects of the volume fraction index n on the critical bucking pressures of the shells.

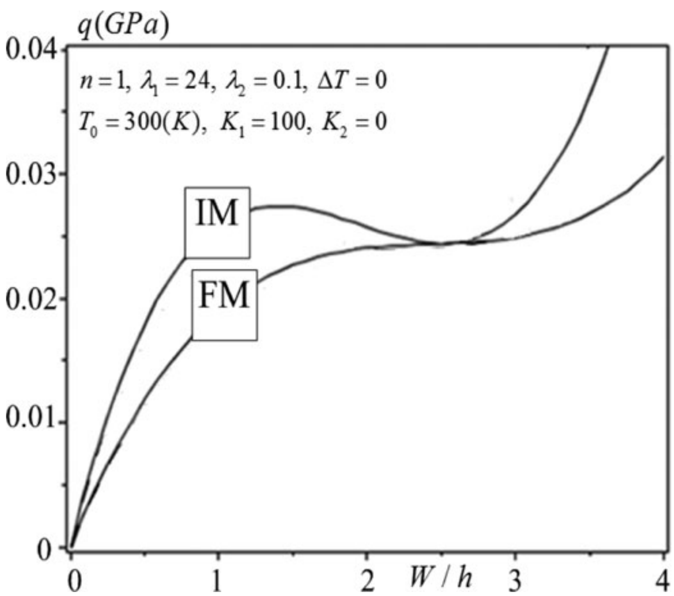

Figure 15. The effects of in-plane restraint conditions on the nonlinear response of the shell.

loads in a post-buckling period. However, the snap-through behavior of the shells with IM is very unstable.

5. Concluding remarks

The present article aims to propose an analytical approach to study the problem of nonlinear post-buckling of shear deformable S-FGM axisymmetric spherical shell with temperature-dependent material properties on elastic foundations subjected to mechanical and thermal loads. Based on the first shear deformable theory, the equilibrium and compatibility equations are derived in terms of the shell deflection and the stress function. From these explicit expressions, the nonlinear axisymmetric response of the shell is analyzed and the results are illustrated in graphic form. The results show that the nonlinear response of the S-FGM spherical shell is complex and greatly influenced by the material and geometric parameters and in-plane restraint. The study also reveals the important role of pre-existent temperature conditions on the nonlinear response of S-FGM shallow spherical shell under uniform external pressure.

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