On the Implementation of Chaotic Compressed Sensing for MRI

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***Abstract*—We consider the application of Compressed Sensing (CS) to enhance the acquisition speed in Magnetic Resonance Imaging (MRI). For CS-based MRI, random sampling is often implemented in the k-space and depends on the uniform distribution and energy distribution of MRI images in the k-space. In contrast, we propose a new deterministic sampling method for CS-based MRI using the logistic map, which has good statistical properties and can be easily converted to uniform-like chaotic sequences. Simulation results confirmed that the proposed method is equivalent to state-of-the-art methods in terms of the relative root-mean-square error and the probability of exact reconstruction.**

***Index Terms*—Compressed sensing; Chaotic systems; Magnetic resonance imaging (MRI); Fast image acquisition.**

I. Introduction

Magnetic Resonance Imaging (MRI) is a non-invasive imaging tool which has been used widely in medicine. It is desirable to enhance the acquisition speed in MRI in order to enhance image contrast and resolution, to avoid physiological effects or scanning time on patients, to overcome physical constraints inherent within the MRI scanner, or to meet timing requirements when imaging dynamic structures or processes. The last two decades have seen a rapid development of fast MRI, which improves one or more stages in MRI, including (i) excitation and acquisition manner, (ii) reducing the acquired data, and (iii) reconstruction. With respect to fast acquisition, a new sampling paradigm called Compressed Sensing (CS) [1–3] permits exact reconstruction of MR images from a number of samples under CS is smaller than that under Nyquist sampling (as such, CS is one type of under-sampling), thereby reducing the acquisition time.

CS works with incoherent sensing modality by linearly projecting the acquiring signal onto a special domain, and can be successfully applied to sparse or compressible signals. In fact, since MR images are generally sparse (i.e., wavelet coefficients of brain MR images are sparse), CS has recently been shown, by Lustig *et al.*, to be successfully applied to MRI to achieve fast acquisition [4]– so abbreviated as CS-MRI. In MRI, the acquired data provide complete Fourier (a.k.a., k-space) measurements. Then, image reconstruction can be done in either the image domain or the k-space domain. Therefore, CS-MRI is a special case of CS, in which the CS under-sampling process is performed in the k-space to achieve an number of samples smaller than the number of complete Fourier coefficients. The reconstruction can be done by sparse approximation. Sampling in CS-MRI is performed randomly with a distribution whose density matches the energy distribution of samples in the k-space.

In contrast to the usual application of random sampling in CS, deterministic sampling in CS has certain advantages such as more efficient recovery time, explicit constructions, efficient storage, and tighter recovery bounds [5]. A special type of deterministic sampling in CS that employs the deterministic chaos were considered in [6, 7]. Then, in [8–10], the authors proposed a chaotic CS method for MRI, so abbreviated as CCS-MRI. The main idea of CCS-MRI is to apply sampling with a Gaussian-like chaotic sequence, which is transformed from the logistic map, one kind of chaotic systems. The numerical results show the equivalent performance as if using random sampling in CS-MRI.

As mentioned above, CS-MRI is a special case of CS and the theoretical base for exact reconstruction is shown in [2, 3], the early papers of CS. One of the main results in these papers is that one can reconstruct exactly the signal from the Fourier ensemble, which is obtained by randomly sampling rows from the orthonormal $N \times N$ Fourier matrix. With the sampling that is based on the Gaussian-like chaotic sequence as in [8–10], the numerical results show the equivalent performance to random sampling, whereas the sampling based on the Gaussian-like chaotic sequence may be not suitable when the energy distribution in the k-space has changed. In the latter case, the sampling need to be changed, and the uniform-like chaotic sequence may give us more flexibility. The logistic map has good statistic properties, and we can generate the uniform-like chaotic sequence based on the "truly" Bernoulli sequence from the logistic map [11, 12]. In this paper, we proposed a new method for CCS-MRI that is based on the logistic map to generate the uniform-like chaotic sequence.

The paper is organized as follows. Section II describes the basic principles of CS-MRI. The proposed CCS-MRI method is then presented in Section III. The performance of the proposed method is assessed in Section IV.

II. Basic Principles of CS-MRI

A. Compressed Sensing Fundamentals

The discrete-to-discrete formulation of CS is briefly described as follows. Suppose that $\mathbf{x} \in \mathbb{R}^N$ is the sparse signal. It means that there exists a vector $\boldsymbol{\alpha}$ containing exactly $K \ll N$ nonzero values, so called the K-sparse vector, and a proper basis $\boldsymbol{\Psi} = [\psi_1, \ldots, \psi_N]$ satisfying $\mathbf{x} = \boldsymbol{\Psi}\boldsymbol{\alpha}$. In this case, the transform matrix $\boldsymbol{\Psi}$, used to represent $\mathbf{x}$ in the sparsity basis, is called the sparsifying matrix or representation basis and $\mathbf{x}$ is referred to as the K-sparse signal. In CS, $\mathbf{x}$ is acquired linearly by measurement vector $\mathbf{y}$ as

$$\mathbf{y} = \boldsymbol{\Phi}\mathbf{x} = \boldsymbol{\Phi}\boldsymbol{\Psi}\boldsymbol{\alpha} = \boldsymbol{\Theta}\boldsymbol{\alpha}, \tag{1}$$

where $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$ is the sensing matrix, $M \ll N$. This matrix $\boldsymbol{\Phi}$ is designed to have M as small as possible while allowing exact reconstruction of $\mathbf{x}$ (or $\boldsymbol{\alpha}$) from the measurement vector $\mathbf{y}$.

Two principles that enable CS are sparsity and incoherence. With supposition that $\mathbf{x}$ is the sparse signal, the sparsity principle is satisfied. Accurate reconstruction $\boldsymbol{\alpha}$ can be achieved when $\boldsymbol{\Theta}$ satisfies the so-called Restricted Isometry Property (RIP) of order K. In other words, $\boldsymbol{\Theta}$ approximately preserves the length of K-sparse signals; all subsets of K columns of $\boldsymbol{\Theta}$ are near orthogonal. One way to satisfy RIP is to have incoherence between $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$: random matrices are largely incoherent with any fixed basis $\boldsymbol{\Psi}$.

When $\boldsymbol{\Theta}$ satisfies RIP, exact reconstruction of $\boldsymbol{\alpha}$ (or, essentially, $\mathbf{x}$ since $\boldsymbol{\Psi}$ is known) is achieved with overwhelming probability using the following ℓ_1 minimization problem:

$$\boldsymbol{\alpha} = \arg\min_{\boldsymbol{\alpha}'} \|\boldsymbol{\alpha}'\|_1 \qquad \text{subject to} \quad \boldsymbol{\Theta}\boldsymbol{\alpha}' = \mathbf{y}. \tag{2}$$

When there is noise in the measurements, the optimization problem is reformulated as follows:

$$\boldsymbol{\alpha} = \arg\min_{\boldsymbol{\alpha}'} \|\boldsymbol{\alpha}'\|_1 \quad \text{subject to} \quad \|\boldsymbol{\Theta}\boldsymbol{\alpha}' - \mathbf{y}\|_2 < \epsilon, \tag{3}$$

where ϵ is a constant related to the variance of the noise.

B. 2D-MRI acquisition

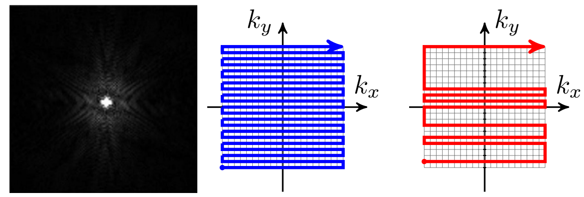
Consider the imaging of a 2D slice of the object in the 2D plane $\{x, y\}$. Let $m(x, y)$ be this image. The analog signal acquired by the receiving coil is represented in the k-space, by the following imaging equation:

$$\nu(k_x, k_y) = \iint_{xy} m(x, y)\, e^{-i(k_x x + k_y y)} dx\, dy, \tag{4}$$

where k_x and k_y encode the information of the locations along x and y directions of the image respectively, and $k = \{k_x, k_y\}$ is said to lie in the k-space. Clearly, the image $m(x, y)$ can be obtained by applying a 2D-Fourier transform on $\nu(k_x, k_y)$. Note that, the time dimension is implicitly included in k_x and k_y.

Upon sampling the k-space, we have the discrete version of (4) as follows:

$$\nu(k_x, k_y) = \sum_{n_x}^{N_x - 1} \sum_{n_y}^{N_y - 1} m(n_x, n_y)\, e^{-i(k_x x + k_y y)} dx dy, \tag{5}$$



(a) (b) (c)

Fig. 1. k-space of a brain MR image. (a)– analog acquisition, (b)– linear sampling, (c) linear undersampling. In (c), a binary mask (of 256×256 points) is applied to (a), followed by a power decay law along direction k_y.

where N_x and N_y are the numbers of pixels along x and y axes of the image. Equation (5) can be expressed in matrix form as

$$\boldsymbol{\nu} = \mathbf{F}\mathbf{m}, \tag{6}$$

There are various ways of defining k_x and k_y, depending on the k-space trajectory in use. A common trajectory for 2D imaging is the Cartesian trajectory used in this paper. Fig. 1(a) presents the k-space of an MR image of a brain slice, which is shown in Fig. 2(a). We see that most of the encoded information concentrates around the origin of the k-space. In practice, the energy distribution in the k-space follows a power law.

C. CS-MRI with Gaussian random measurements

In CS-MRI, the imaging equation results in incomplete measurements in the k-space in the form of the measurement vector

$$\boldsymbol{\nu} = \mathbf{P}\mathbf{F}\mathbf{m}, \tag{7}$$

where $\mathbf{P} \in \mathbb{R}^{M \times N}$ is a rectangular binary matrix containing only one non-zero value on each row, representing the action of randomly selecting only M rows out of N rows of $\mathbf{F}$. By corresponding the CS model in (1) and the imaging equation in standard MRI in (7), one can see that the CS measurements $\mathbf{y} \equiv \boldsymbol{\nu}$, the measurement matrix $\boldsymbol{\Phi} \equiv \mathbf{PF}$ and the underlying signal to be reconstructed $\mathbf{x} \equiv \mathbf{m}$.

We have just mentioned that the energy distribution in the k-space follows a power law so, for undersampling in the k-space, one can use either Gaussian random measurements, or combination of uniform random measurements and the density matching with the energy distribution in the k-space. The number of k-space samples is much smaller than that obtained by linear (full) sampling as described above. MRI reconstruction from the k-space samples is performed by Non-linear Conjugate Gradient (NCG) [4]. Suppose the image of interest is a vector $\mathbf{m}$. The reconstructed image is obtained by solving the following constrained optimization problem:

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} \left\{ \|\mathbf{PFm} - \boldsymbol{\nu}\|_2^2 + \lambda \|\boldsymbol{\Psi}\mathbf{m}\|_1 \right\}, \tag{8}$$

where λ is a tuning constant for the trade-off between fidelity term and the sparsity, ϵ controls the fidelity term, and $\boldsymbol{\Psi}$ represents the sparsifying matrix in the wavelet domain.

III. Proposed Method for CCS-MRI

A. Principle of CCS-MRI

CS-MRI constructs the matrix $\mathbf{P}$ in Equation (7) randomly, this corresponds to randomly select M rows of the Fourier coefficient matrix compatible with the energy distribution in the k-space. The sampling process of CCS-MRI, in contrast, depends on chaos which are deterministic but behaves like Gaussian or uniform distribution. Now we consider CCS-MRI in detail.

To construct the chaotic measurement matrix $\mathbf{\Phi}$, firstly, a logistic map is created by [13]:

$$h_L(n+1) = \alpha h_L(n)(1 - h_L(n)), \tag{9}$$

where α is a control parameter and $n = 0, ..., L-1$. Note that choosing the initial condition $h_L(0)$ is very sensitive to formulate a suitable chaotic sequence. The logistic map with $\alpha = 4$ is fully chaotic such that for any initial condition almost every point on the unit interval is visited and the probability distribution function of the output is symmetric. Consequently, the logistic map is transformed to Gaussian-logistic map in order to make the chaotic sequence behave Gaussian-like, by the following conversion:

$$G_L(n) = \ln\left(\frac{h_L(n)}{1 - h_L(n)}\right). \tag{10}$$

The indices of selected rows are specified by the values of $G_L(n)$.

B. Proposed method for CCS-MRI

We propose in this section a new method, called NewCCS-MRI, for CCS-MRI based on the logistic map. The proposed method can be summarized as follows:

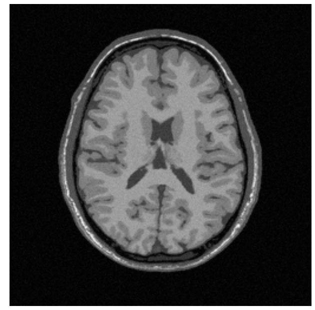
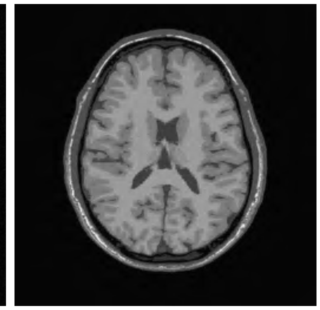
1) Generate the logistic sequence according to (9).
2) Transform the logistic sequence to a binary sequence by

$$B(n) = \sigma(h_L(n)) = \begin{cases} 1, & \text{if } h_L(n) > \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

   The results in [12] show that $B(n)$ is "truly" binary Bernoulli sequence, i.e., for any $q > 0$, with $\mathbf{b} = (b_1, b_2, \ldots, b_q)$, $b_i \in \{0, 1\}$ for $i = 1, 2, 3, \ldots, q$ we have

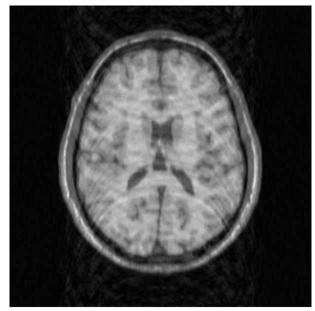
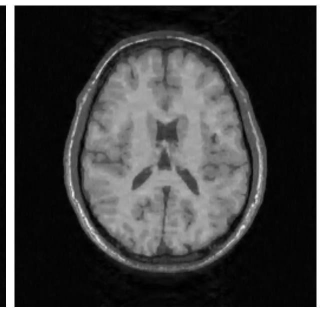
$$\text{Prob}\,(\mathbf{b}, \sigma) = 0.5^q, \tag{12}$$

   where $\text{Prob}\,(\mathbf{b}, \sigma)$ denotes the probability of the event $\sigma(h_L(1)) = b_1, \sigma(h_L(2)) = b_2, \ldots, \sigma(h_L(q)) = b_q$.
3) To sample Fourier coefficient matrix with size of $N \times N$, where N is usually set to 2^b, we then convert series of b bits to integer. With this transformation, the truly binary Bernoulli sequence is converted to a uniform sequence. The indices of the selected rows are specified by the values of this sequence and the number of selected rows M satisfies $M \geq \text{Const}K \log(N)$, with K being a constant relative to the sparse of the image (i.e., the considered signal is K-sparse).

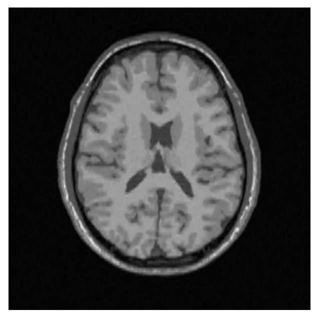
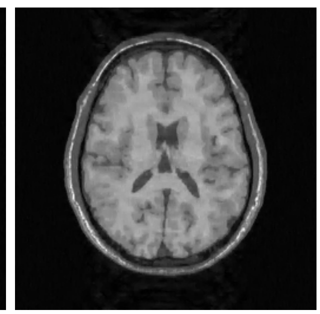
(a) The original image

(b) Reconstruction from largest 10% wavelet coefficients

(c) Reconstruction by zero filling

(d) Reconstruction from CS-MRI

(e) Reconstruction from CCS-MRI

(f) Reconstruction from NewCCS-MRI

Fig. 2. Original brain slice image and reconstructed images from largest 10% wavelet coefficients, CS-MRI, CCS-MRI and NewCCS-MRI at $r = 0.35$.

IV. Simulation

We will assess the performance of the proposed method by qualitative and quantitative evaluations of image reconstruction.

We use an MR image of size 256×256 pixels as shown in Fig. 2(a). The logistic map is implemented as described above with $\alpha = 4$ and the initial condition $h_L(0) = 3$. For CS-MRI and NewCCS-MRI, the indices of the selected rows are determined by the values of the random sequence or the uniform-like chaotic sequence and the sampling density scaling according to a power of distance from the origin; the density power is 3. The simulation is done using SparseMRI[1] software [4].

For each reconstruction, the relative root-mean-square error

[1] SparseMRI: http://people.eecs.berkeley.edu/~mlustig/Software.html.

(RMSE) is given by

$$\mathrm{RMSE} = \sqrt{\frac{\|\mathbf{m} - \hat{\mathbf{m}}\|^2}{\|\mathbf{m}\|^2}}, \tag{13}$$

where $\mathbf{m}$ and $\hat{\mathbf{m}}$, respectively, denote the reconstructed image from full sampling and Compressed Sensing for MRI. We also want to assess the RMSE with respect to the compression ratio, defined as $r = M/N$.

Fig. 2(b) shows the reconstructed image from 10% wavelet coefficients and the RMSE is 0.0582; the reconstructed image when the k-space was under-sampled and inverse Fourier transform was used for reconstruction is illustrated in Fig. 2(c). The ringing artefact in this image reflects the aliasing effect due to under-sampling. Figs. 2(d), 2(e) and 2(f) compare the reconstructed images obtained from CS-MRI, CCS-MRI and NewCCS-MRI, respectively (for $r = 0.35$). We can see that the quality of the reconstructed image quality by NewCCS-MRI is as good as that by CS-MRI or CCS-MRI.

We further examine the effect of compression ratio with respect to the RMSE (Fig. 4). At low compression ratios (i.e., more compressed), NewCCS-MRI and CS-MRI have better performance (i.e., lower RMSE) than CCS-MRI. At high compression ratios, in contrast, CCS-MRI is superior to the others. Moreover, performance of NewCCS-MRI is the same as that of CS-MRI.

Finally, a performance study based on the probability of exact reconstruction (recovery rate) is illustrated in Fig. 4. The decision for failure reconstruction is made when the relative RMSE of reconstruction is greater than 0.0582. Note that the value of 0.0582 is the relative RMSE of reconstructed image from 10% wavelet coefficients. However, for this method, we have to collect full observation of the k-space. The probability of exact reconstruction reconfirms the "equivalence" of CS-MRI and NewCCS-MRI. The recovery rate depends on error metric and decision threshold for failure or success in image reconstruction. At the ratio $r = 0.35$ as we have seen in Fig. 2, although a "failure" in reconstruction occurred, the image quality is comparable to the original image.

V. Conclusions

This paper presents a new method to enhance the speed of acquisition for MRI applications. For the sake of simplicity in presenting the idea of CS, this paper only considered 2D standard MRI. With this presentation, instead of fully sampling the k-space, we only selected M out of N horizontal trajectories in the k-space and the selection of these trajectories was done using the values generated from the chaotic sequence (CCS-MRI and NewCCS-MRI). The good performance of CCS-MRI and NewCCS-MRI at values of the compression ratio r from 3.5 to 0.5 implies that the speed of acquisition in MRI can be enhanced by a factor from 2 to 2.86 when we apply the deterministic sampling methods based on chaos for CS. Finally, with the NewCCS-MRI, we can change the sampling density matching with the energy distribution in the k-space more flexibly than other sampling methods which depend on the Gaussian-like chaotic sequence.

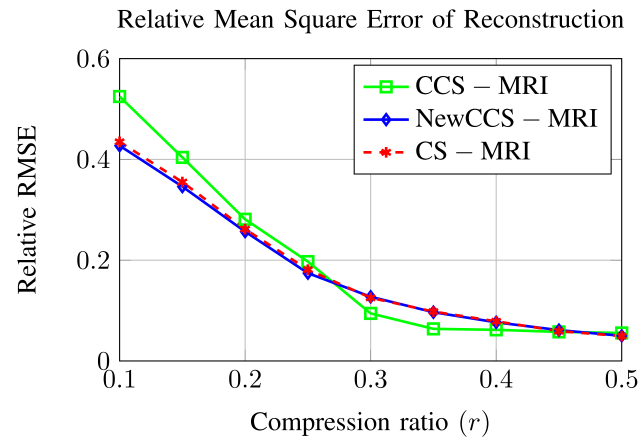


Fig. 3. Relative RMSE of reconstruction.

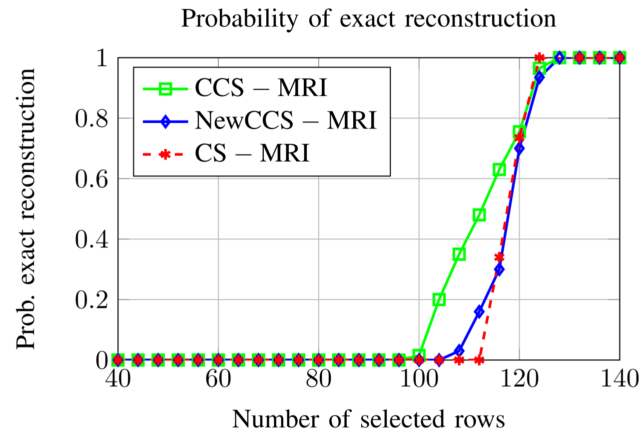


Fig. 4. Probability of exact reconstruction with the number of selected rows from 40 lines to 140 lines (r from 0.16 to 0.55).

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