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# Nonlinear axisymmetric response of FGM shallow spherical shells on elastic foundations under uniform external pressure and temperature

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ABSTRACT

Based on the classical shell theory taking into account geometrical nonlinearity, initial geometrical imperfection and Pasternak type elastic foundation, the nonlinear axisymmetric response of shallow spherical FGM shells under mechanical, thermal loads and different boundary conditions is considered in this paper. Using the Bubnov–Galerkin method and stress function, obtained results show effects of elastic foundations, external pressure, temperature, material and geometrical properties on the nonlinear buckling and postbuckling of the shells. The snap-through behaviors of the FGM spherical shallow shells on elastic foundations also are analyzed carefully in this paper. Some results were compared with the ones of other authors.



## 1. Introduction

The spherical shells play an important role in the engineering application. For example, they have been used to make several items found on the aircrafts, the spaceship as well as the shipbuilding industry and the civil engineering. Hence, the problems associated with the behavior of the spherical FGM shells buckling and postbuckling have received much interest in the recent years.

Functionally Graded Materials (FGMs), which are consisting of metal and ceramic constituents, is one class of these structures. Due to intelligent characteristics such as high stiffness, excellent thermal resistance capacity, FGMs are now chosen to use as structural constituents exposed to severe temperature conditions such as aircraft, aerospace structures, nuclear plants and other engineering applications. Unfortunately, there is a subtle understanding of the spherical FGM shell due to the difficulties in a calculation. Indeed, there are not many studies on this problem. Tillman (1970) investigated the buckling behavior of shallow spherical caps under a uniform pressure load. Nath and Alwar (1978) analyzed non-linear static and dynamic response of spherical shells. Buckling and postbuckling behavior of laminated shallow spherical shells subjected to external pressure been analyzed by Muc (1992) and Xu (1991). Alwar and Narasimhan (1992) used method of global interior collocation to study axisymmetric nonlinear behavior of laminated orthotropic annular spherical shells. Ganapathi (2007) studied dynamic stability characteristics of functionally graded materials shallow spherical shells using the first order shear deformation theory and finite element method. On the nonlinear axisymmetric dynamic buckling behavior of clamped functionally graded spherical caps been analyzed by Prakash et al. (2007). Bich (2009) has been credited for the first calculation of the nonlinear buckling of FGM shallow spherical shells. In his investigation, he has used analytical approach taken into account the geometrical nonlinearity. Recently, Bich and Hoa (2010, 2011, 2012) has developed the nonlinear static and dynamic for FGM shallow spherical shells subjected to the mechanical and thermal loads.

The structures widely used in aircraft, reusable space transportation vehicles and civil engineering are usually supported by an elastic foundation. Therefore, it is necessary to include effects of elastic foundation for a better understanding of the buckling behavior and loading carrying capacity of plates and shells. Librescu and his co-workers have investigated the postbuckling behavior of flat and curved laminated composite shells resting on Winkler elastic foundations (Librescu and Lin, 1997; Lin and Librescu, 1998). Huang et al. (2008) proposed solutions for functionally graded thick plates resting on Winkler–Pasternak elastic foundations. Shen (2009) and Shen et al. (2010) investigated the postbuckling behavior of FGM cylindrical shells subjected to axial compressive loads and internal pressure and surrounded by an elastic medium of the Pasternak type. Duc extend his investigations for nonlinear dynamic response of imperfect eccentrically stiffened FGM double curved shallow shells on elastic foundation (Duc, 2013). In spite of practical importance and

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increasing use of FGM structures, investigations on the effects of elastic media on the response of FGM plates and shells are comparatively scarce. To best of authors' knowledge, there is no analytical investigation on the nonlinear stability of FGM shallow spherical shells on elastic foundation.

In this paper, we have made a further investigation of FGM spherical shell for which Dumir (1985) have studied the nonlinear axisymmetric response of orthotropic thin spherical caps on elastic foundation. Nie (2001) proposed the asymptotic iteration method to treat nonlinear buckling of externally pressurized isotropic shallow spherical shells with various boundary conditions incorporating the effects of imperfection, edge elastic restraint and elastic foundation.

In the paper, we consider the nonlinear axisymmetric buckling and postbuckling of the shallow spherical FGM shells on elastic foundation using classical shell theory (CST) taking into account geometrical nonlinearity and initial geometrical imperfection. The properties of materials are graded in thickness direction according to a power law function of thickness coordinate. Two cases of thermal loads are considered: uniform temperature rise and through the thickness temperature gradient. Using the Bubnov–Galerkin method and stress function, obtained results show effects of external pressure, temperature, material and geometrical properties, imperfection and elastic foundation on the nonlinear response of clamped shallow spherical shells.

## 2. Theoretical formulations

### 2.1. Functionally graded shallow spherical shells on elastic foundation

We consider a FGM shallow spherical shell resting on elastic foundations with radius of curvature $R$, base radius $r_0$ and thickness $h$ in coordinate system $(\varphi, \theta, z)$, $-h/2 \le z \le h/2$ as shown in Fig. 1.

The effective properties of FGM shallow spherical shell such as modulus of elasticity $E$, the coefficient of thermal expansion $\alpha$, the coefficient of thermal conduction $K$, and the Poisson ratio $\nu$ is assumed constant can be defined as (Bich and Tung, 2011; Duc, 2013)

$$[E(z), \alpha(z), K(z)] = [E_m, \alpha_m, K_m] + [E_{cm}, \alpha_{cm}, K_{cm}]\left(\frac{2z+h}{2h}\right)^N;$$
$$\nu(z) = \nu = \text{const} \tag{1}$$

where $N \ge 0$ is volume fraction index and $E_{cm} = E_c - E_m$, $\alpha_{cm} = \alpha_c - \alpha_m$, $K_{cm} = K_c - K_m$. The subscripts m and c stand for the metal and ceramic constituents, respectively.

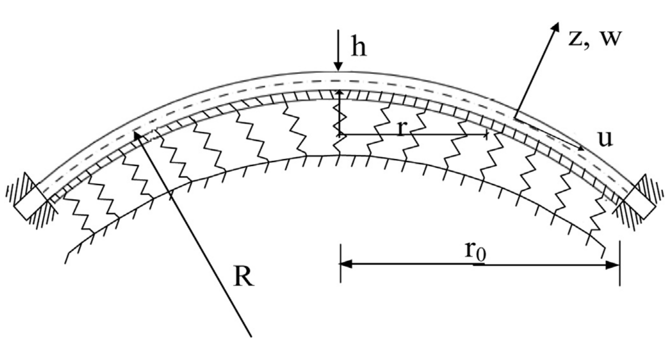


**Fig. 1.** FGM shallow spherical shell on elastic foundation.

It is evident that $E = E_c$, $\alpha = \alpha_c$, $K = K_c$ at $z = h/2$ (surface is ceramic-rich) and $E = E_m$, $\alpha = \alpha_m$, $K = K_m$ at $z = -h/2$ (surface is metal-rich).

Note that the case when the Poisson ratio is varied smoothly along the thickness $\nu = \nu(z)$ has considered by Huang and Han (2008, 2010), Duc and Quan (2012, 2013), Cong (2011), Duc (2013). The obtained results show that effects of Poisson's ratio $\nu$ is very small. Therefore, for simplicity, as well as many other authors, in this paper we assumed $\nu = $ const.

The above elastic foundations are simply described by a load which can be written in the following form (Shen et al., 2010; Duc, 2013; Duc and Quan, 2013):

$$q_e = k_1 w - k_2 \Delta w \tag{2}$$

where $\Delta w = w_{,rr} + 1/r w_{,r} + 1/r^2 w_{,\theta\theta}$, $w$ is the deflection of the shallow spherical shell, $k_1$ is Winkler foundation modulus and $k_2$ is the shear layer foundation stiffness of Pasternak model.

### 2.2. Governing equations

The theory of the classical thin shells has been applied to investigate the non-linear stability of the shallow spherical FGM. For simplicity, we have introduced the variable $r = R\sin\varphi$ which is indeed a radius of the circle. For a shallow spherical case, we can use an approximation $\cos\varphi = 1$ and $Rd\varphi = dr$.

The deformation factors of a spherical shell at a distance $z$ with respect to the central surface can be determined as the follows:

$$\varepsilon_r = \varepsilon_r^0 + z\chi_r; \varepsilon_\theta = \varepsilon_\theta^0 + z\chi_\theta; \gamma_{r\theta} = \gamma_{r\theta}^0 + 2z\chi_{r\theta} \tag{3}$$

where $\varepsilon_r^0$ and $\varepsilon_\theta^0$ are the normal strains, $\gamma_{r\theta}^0$ is the shear strain at the middle surface of the spherical shell and $\chi_r$, $\chi_\theta$ are curvatures, $\chi_{r\theta}$ is a twist.

Using CST, we have (Bich and Tung, 2011; Bich et al., 2012; Brush, 1975):

$$\varepsilon_r^0 = u_{,r} - \frac{w}{R} + \frac{1}{2}w_{,r}^2; \varepsilon_\theta^0 = \frac{v_{,\theta} + u}{r} - \frac{w}{R} + \frac{1}{2r^2}w_{,\theta}^2; \gamma_{r\theta}^0$$
$$= r\left(\frac{v}{r}\right)_{,r} + \frac{u_\theta}{r} + \frac{1}{r}w_{,r}w_{,\theta} \tag{4}$$

$$\chi_r = -w_{,rr}; \chi_\theta = -\frac{w_{,\theta\theta}}{r^2} - \frac{w_{,r}}{r}; \chi_{r\theta} = -\frac{1}{r}w_{,r\theta} + \frac{w_{,\theta}}{r^2} \tag{5}$$

For a spherical shell, Hooke's law which describes the relationship between the stress and strain in the presence of temperature, is written as:

$$\begin{aligned}(\sigma_r, \sigma_\theta) &= \tfrac{E}{1-\nu^2}[(\varepsilon_r, \varepsilon_\theta) + \nu(\varepsilon_\theta, \varepsilon_r) - (1+\nu)\alpha\Delta T(1,1)] \\ \sigma_{r\theta} &= \tfrac{E}{2(1+\nu)}\gamma_{r\theta}\end{aligned} \tag{6}$$

where $\Delta T$ is augmenter of temperature between the surfaces of the shell.

The internal force as well as the moment inside the spherical shell FGM can be determined as:

$$(N_i, M_i) = \int_{-h/2}^{h/2} \sigma_i(1, z)dz, \quad i = r, \theta, r\theta \tag{7}$$

We substitute (1) and (3) into (6), then insert the derived result to (7), we finally come up with the internal force and the moment's constituents as the follows:

$$\begin{aligned}
[N_r, M_r] &= \tfrac{[E_1,E_2]}{1-\nu^2}\left(\varepsilon_r^0 + \nu\varepsilon_\theta^0\right) + \tfrac{[E_2,E_3]}{1-\nu^2}(\chi_r + \upsilon\chi_\theta) - \tfrac{[\Phi_a,\Phi_b]}{1-\nu} \\
[N_\theta, M_\theta] &= \tfrac{[E_1,E_2]}{1-\nu^2}\left(\varepsilon_\theta^0 + \nu\varepsilon_r^0\right) + \tfrac{[E_2,E_3]}{1-\nu^2}(\chi_\theta + \upsilon\chi_r) - \tfrac{[\Phi_a,\Phi_b]}{1-\nu} \\
[N_{r\theta}, M_{r\theta}] &= \tfrac{[E_1,E_2]}{2(1+\upsilon)}\gamma_{r\theta}^0 + \tfrac{[E_2,E_3]}{1+\upsilon}\chi_{r\theta}
\end{aligned} \tag{8a}$$

$$(\Phi_a, \Phi_b) = \int_{-h/2}^{h/2} (1, z)E(z)\alpha(z)\Delta T(z)dz \tag{8b}$$

the explicit analytical expressions of $E_i (i = 1–3)$ are calculated and given in the Appendix A.

The equilibrium equations of a perfect spherical shell under external pressure $q$ and resting on elastic foundations are given by Bich and Tung (2011), Bich et al. (2012) and Brush (1975):

$$\begin{aligned}
(rN_r)_{,r} + N_{r\theta,\theta} - N_\theta &= 0 \\
(rN_{r\theta})_{,r} + N_{\theta,\theta} + N_{r\theta} &= 0
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{r}\left[(rM_r)_{,rr} + 2\left(M_{r\theta,r\theta} + \frac{1}{r}M_{r\theta,\theta}\right) + \frac{1}{r}M_{\theta,\theta\theta} - M_{\theta,r}\right] \\
&+\frac{1}{R}(N_r + N_\theta) + \frac{1}{r}\left(rN_r w_{,r} + N_{r\theta}w_{,\theta}\right)_{,r} + \frac{1}{r}\left(N_{r\theta}w_{,r} + \frac{1}{r}N_\theta w_{,\theta}\right)_{,\theta} \\
&+q - k_1 w + k_2 \Delta w = 0
\end{aligned} \tag{9}$$

The first two equations in the set of equilibrium equation (9) have been satisfied simultaneously if we introduce the stress function $f(r, \theta)$ under the following conditions:

$$N_r = \frac{1}{r}f_{,r} + \frac{1}{r^2}f_{,\theta\theta}, N_\theta = f_{,rr}, N_{r\theta} = -\left(\frac{f_\theta}{r}\right)_{,r} \tag{10}$$

Insert Eqs. (5), (8a) and (10) into the third equation in (9), we have:

$$\begin{aligned}
&D\Delta^2 w - \frac{1}{R}\Delta f - \left(\frac{1}{r}f_{,r} + \frac{1}{r^2}f_{,\theta\theta}\right)w_{,rr} + \frac{2}{r}\left(\frac{f_{\theta r}}{r} - \frac{f_\theta}{r^2}\right)w_{,\theta r} - \frac{w_{,r}f_{,rr}}{r} \\
&+\left(\frac{f_{,\theta}}{r^2} - \frac{f_{,\theta r}}{r}\right)\frac{2}{r^2}w_{,\theta} - \frac{1}{r^2}w_{,\theta\theta}f_{,rr} - q + k_1 w - k_2\Delta w = 0
\end{aligned} \tag{11}$$

where

$$D = \frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)}, \Delta() = ()_{,rr} + \frac{1}{r}()_{,r} + \frac{1}{r^2}()_{,\theta\theta} \tag{12}$$

Eq. (11) is the equilibrium equation of a spherical shell derived from two functions which are the bending function $w$ and stress function $f$. In order to derive the function which combines these two functions, we can apply the following compatibility equation:

$$\frac{1}{r^2}\varepsilon^0_{r,\theta\theta} - \frac{1}{r}\varepsilon^0_{r,r} + \frac{1}{r^2}\left(r^2\varepsilon^0_{\theta,r}\right)_{,r} - \frac{1}{r^2}\left(r\gamma^0_{r\theta}\right)_{,r\theta} = -\frac{\Delta w}{R} + \chi^2_{r\theta} - \chi_r\chi_\theta \tag{13}$$

From Eqs. (5) and (8a), we can calculate $\varepsilon_\theta^0, \varepsilon_r^0, \gamma_{r\theta}^0$ as the follows:

$$\begin{aligned}
\varepsilon_\theta^0 &= \tfrac{N_\theta - \nu N_r}{E_1} + \tfrac{E_2}{E_1}\left(\tfrac{w_{,\theta\theta}}{r^2} + \tfrac{w_{,r}}{r}\right) + \Phi_a \\
\varepsilon_r^0 &= \tfrac{N_r - \nu N_\theta}{E_1} + \tfrac{E_2 w_{,rr}}{E_1} + \Phi_a \\
\gamma_{r\theta}^0 &= N_{r\theta}\tfrac{2(1+\nu)}{E_1} - \tfrac{2E_2}{E_1}\left(-\tfrac{1}{r}w_{,r\theta} + \tfrac{w_{,\theta}}{r^2}\right)
\end{aligned} \tag{14}$$

Setting Eqs. (14) and (10) into Eq. (13) gives the compatibility equation of a perfect FGM shallow spherical shell as (Bich and Tung, 2011):

$$\frac{1}{E_1}\Delta^2 f = -\frac{\Delta w}{R} + \left(\frac{1}{r}w_{,r\theta} - \frac{1}{r^2}w_\theta\right)^2 - w_{,rr}\left(\frac{1}{r^2}w_{,\theta\theta} + \frac{1}{r}w_{,r}\right) \tag{15}$$

Eqs. (11) and (15) are nonlinear equilibrium and compatibility equations in terms of variables $w$ and $f$ and used to investigate the buckling and postbuckling of FGM shallow spherical shell resting on elastic foundations with asymmetric deformation.

In particular, we apply Eqs. (11) and (15) for the axially symmetric shallow spherical shell (Bich and Tung, 2011; Huang, 1964), we get the equilibrium and compatibility equations written as:

$$D\Delta_s^2 w - \frac{1}{R}\Delta_s f - \frac{1}{r}f_{,r}w_{,rr} - \frac{w_{,r}f_{,rr}}{r} - q + k_1 w - k_2\Delta_s w = 0 \tag{16}$$

$$\frac{1}{E_1}\Delta_s^2 f = -\frac{\Delta_s w}{R} - \frac{1}{r}w_{,rr}w_{,r} \tag{17}$$

For a perfect case of the axially symmetric shallow spherical shell, where $\Delta_s() = ()'' + ()'/r$ and prime indicates differentiation with respect to $r$, i.e. $()' = d()/dr$.

For an imperfect FGM spherical shell, Eqs. (16) and (17) are modified into forms as

$$\begin{aligned}
&D\Delta_s^2 w - \frac{1}{R}\Delta_s f - \frac{1}{r}f_{,r}\left(w_{,rr} + w^*_{,rr}\right) - \frac{f_{,rr}}{r}\left(w_{,r} + w^*_{,r}\right) - q \\
&+ k_1 w - k_2\Delta_s w = 0
\end{aligned} \tag{18}$$

$$\frac{1}{E_1}\Delta_s^2 f = -\frac{\Delta_s w}{R} - \frac{1}{r}w_{,rr}w_{,r} - \frac{1}{r}w_{,r}w^*_{,rr} - \frac{1}{r}w_{,rr}w^*_{,r} \tag{19}$$

in which $w^*(r)$ is a known function representing initial small imperfection of the shell.

Eqs. (18) and (19) are nonlinear governing equations in terms of variables $w$ and $f$ and used to investigate the buckling and postbuckling of an imperfect FGM spherical shell resting on elastic foundations and subjected to mechanical, thermal and thermo-mechanical loads.

## 3. Nonlinear stability analysis

In this paper, two cases of boundary conditions will be considered (Uemura, 1971; Li et al., 2003):

Case (1). The edges are clamped and freely movable (FM) in the meridional direction. The associated boundary conditions are

$$\begin{aligned}
r &= 0, w = W, w' = 0 \\
r &= r_0, w = w' = 0, N_r = 0
\end{aligned} \tag{20}$$

Case (2). The edges are clamped and immovable (IM). For this case, the boundary conditions are

$$\begin{aligned}
r &= 0, w = W, w' = 0 \\
r &= r_0, w = w' = 0, N_r = N_{r0}
\end{aligned} \tag{21}$$

where $W$ is the largest bending and $N_{r0}$ is the normal force on the edge.

The approximation root has been chosen to satisfy the boundary conditions (20) and (21):

$$w = W\frac{\left(r_0^2 - r^2\right)^2}{r_0^4} \tag{22}$$

$$w^* = \mu h \frac{(r_0^2 - r^2)^2}{r_0^4} \tag{23}$$

Where the imperfect function of the spherical shell has been assumed to have the same form as the bending function in which $\mu$ contributes to the imperfection (i.e. $-1 \le \mu \le 1$) (Bich and Tung, 2011; Huang, 1964).

Replacing the Eqs. (22) and (23) into (19) and we then integrate the final equation, we have

$$f' = -\frac{E_1 W}{r_0^4 R}\left(\frac{r^5}{6} - \frac{r_0^2 r^3}{2}\right) - \frac{E_1 W(W + 2\mu h)}{r_0^8}\left(\frac{r^7}{6} - \frac{2r_0^2 r^5}{3} + r_0^4 r^3\right) + \frac{C_1 r}{4}(2\ln r - 1) + \frac{C_2 r}{2} + \frac{C_3}{r} \tag{24}$$

where $C_1$, $C_2$, $C_3$ are the integral constants. Since the deformation as well as the internal force at the top of the spherical shell are limited, $r = 0$, the constants $C_1$ and $C_3$ are zero. The boundary condition $N_r(r = a) = N_{r0}$, gives us the constant $C_2$. The stress function $f$ has been determined as the follows:

$$f' = -\frac{E_1 W}{r_0^4 R}\left(\frac{r^5}{6} - \frac{r_0^2 r^3}{2}\right) - \frac{E_1 W(W + 2\mu h)}{r_0^8}\left(\frac{r^7}{6} - \frac{2r_0^2 r^5}{3} + r_0^4 r^3\right) - \frac{E_1 W}{3R} r + \frac{E_1 W(W + 2\mu h)}{2r_0^2} r + N_{r0} r \tag{25}$$

In case of $N_{r0} = 0$ for the mobile edge of the spherical shell.

Substituting Eqs. (22), (23) and (25) into Eq. (18) and applying Bubnov–Galerkin method for the resulting equation yield

$$q = \left(\frac{64D}{r_0^4} + \frac{3E_1}{7R^2}\right)W - \frac{976E_1}{693r_0^2 R}W(W + \mu h) - \frac{409E_1}{693Rr_0^2}W(W + 2\mu h) + \frac{848E_1}{429r_0^4}(W + \mu h)W(W + 2\mu h) + \frac{40}{7r_0^2}N_{r0}(W + \mu h) - \frac{2N_{r0}}{R} + k_1\frac{16}{21}W + \frac{40}{7r_0^2}k_2 W \tag{26}$$

Eq. (26) is governing equations used to investigate the nonlinear static axisymmetric buckling of clamped FGM shallow spherical shells on elastic foundations under uniform external pressure and thermal loads.

### 3.1. Nonlinear mechanical stability analysis

The shell is assumed to be subjected to external pressure $q$ uniformly distributed on the outer surface of the shell with FM edge (Case (1)). In this case $N_{r0} = 0$ and Eq. (26) gives

$$q = b_1^1\overline{W} - b_2^1\overline{W}(\overline{W} + \mu) - b_3^1\overline{W}(\overline{W} + 2\mu) + b_4^1\overline{W}(\overline{W} + \mu) \times (\overline{W} + 2\mu) \tag{27}$$

The explicit analytical expressions of $b_i^1 (i = 1 - 4)$ are calculated and given in the Appendix A.

If FGM spherical shell does not rest on elastic foundations ($K_1 = K_2 = 0$), we received:

$$q = \left(\frac{64\overline{D}}{R_0^4 R_h^4} + \frac{3\overline{E_1}}{7R_h^2}\right)\overline{W} - \frac{\overline{E_1}}{693R_0^2 R_h^3}\overline{W}(1385\overline{W} + 1794\mu) + \frac{848\overline{E_1}}{429R_0^4 R_h^4}(\overline{W} + \mu)\overline{W}(\overline{W} + 2\mu) \tag{28}$$

The equation (28) is obtained by Bich and Tung (2011).

For a perfect spherical shell, i.e. $\mu = 0$, it is deduced from Eq. (27) that

$$q = \left(\frac{64\overline{D}}{R_0^4 R_h^4} + \frac{3\overline{E_1}}{7R_h^2} + K_1\frac{16\overline{D}}{21R_0^4 R_h^4} + \frac{40\overline{D}}{7R_0^4 R_h^4}K_2\right)\overline{W} - \frac{1385\overline{E_1}}{693R_0^2 R_h^3}\overline{W}^2 + \frac{848\overline{E_1}}{429R_0^4 R_h^4}\overline{W}^3 \tag{29}$$

For a perfect spherical shells, extremum points of curves $q(\overline{W})$ are obtained from condition:

$$\frac{dq}{d\overline{W}} = A - 2B\overline{W} + C\overline{W}^2 = 0 \tag{30}$$

which yields

$$\overline{W}_{l,u} = \frac{B \pm \sqrt{B^2 - AC}}{C} \tag{31}$$

$$\text{provided} \quad B^2 - AC > 0 \tag{32}$$

It is easy to examine that if condition (32) is satisfied $q(\overline{W})$ curve of the perfect shell reaches minimum at $\overline{W}_l$ and maximum at $\overline{W}_u$ with respective load values are $q_l$ and $q_u$. Here $q_u$, $q_l$ represent respectively upper and lower limit buckling loads of perfect FGM spherical shell under uniform external pressure. The shell will exhibit a snap-through behavior whose intensity is measured by difference between upper and lower buckling loads $\Delta q = q_u - q_l = 4(B^2 - AC)^{3/2}/3C^2$. The explicit analytical expressions of $A$, $B$, $C$ and $q_u$, $q_l$ are calculated and given in the Appendix A.

### 3.2. Nonlinear thermomechanical stability analysis

We consider a clamped FGM spherical shell under external pressure $q$ and thermal load. The condition expressing the immovability on the boundary edge (IM) (Case 2), i.e. $u = 0$ on $r = r_0$, is fulfilled on the average sense as

$$\int_0^{\pi}\int_0^{r_0} \frac{\partial u}{\partial r} r\,dr\,d\theta = 0 \tag{33}$$

From Eqs. (4), (8a) and (10) and we have taken into account the axial symmetry as well as the imperfection. It is easy to show that:

$$\frac{\partial u}{\partial r} = \frac{1}{E_1}\left(\frac{f'}{r} - \nu f''\right) + \frac{E_2}{E_1}w'' - \frac{1}{2}(w')^2 - w'w^{*\prime} + \frac{w}{R} + \frac{\Phi_a}{E_1} \tag{34}$$

Substitute the Eqs. (22), (23) and (25) into (34), then we insert the final result to (33), we get:

$$N_{r0} = -\frac{\Phi_a}{1 - \nu} + \left[\frac{(5\nu - 7)E_1}{36(1 - \nu)R} - \frac{2E_2}{(1 - \nu)r_0^2}\right]W + \frac{(35 - 13\nu)E_1}{72(1 - \nu)r_0^2}W(W + 2\mu h) \tag{35}$$

The Eq. (35) is the normal force on the immobile edge.

Specific expressions of parameter $\Phi_a$ in two cases of thermal loading will be determined.

*3.2.1. Uniform temperature rise*

The FGM spherical shell is exposed to temperature environments uniformly raised from stress free initial state $T_i$ to final value $T_f$ and temperature increment $\Delta T = T_f - T_i$ is considered to be independent from thickness variable.

The thermal parameter $\Phi_a$ can be determined from (8b). Subsequently, employing this expression $\Phi_a$ in Eq. (35) and then substitution of the result into Eq. (26) lead to

$$q = b_1^2\overline{W} + b_2^2\overline{W}(\overline{W}+2\mu) + b_3^2(\overline{W}+\mu)\overline{W}(\overline{W}+2\mu) + b_4^2\overline{W}(\overline{W}+\mu) - \frac{40P\Delta T}{7(1-\nu)R_0^2R_h^2}\overline{W} + \frac{P\Delta T}{1-\nu}\left(\frac{2}{R_h} - \frac{40}{7R_0^2R_h^2}\mu\right) \tag{36}$$

The explicit analytical expressions of $b_i^2 (i = 1\text{–}4)$ and $P$ are calculated and given in the Appendix A.

If the FGM spherical shell does not rest on elastic foundations, the equation (36) coincides with the governing equation given by Bich and Tung (2011).

*3.2.2. Through the thickness temperature gradient*

The metal-rich surface temperature $T_m$ is maintained at stress free initial value while ceramic-rich surface temperature $T_c$ is elevated and nonlinear steady temperature conduction is governed by one-dimensional Fourier equation (Bich and Tung, 2011):

$$\frac{d}{d\overline{z}}\left[K(\overline{z})\frac{dT}{d\overline{z}}\right] + \frac{2K(\overline{z})}{\overline{z}}\frac{dT}{d\overline{z}} = 0;\ T(\overline{z} = R-h/2) = T_m, T(\overline{z} = h+R/2) = T_c \tag{37}$$

In which, $T_c$ and $T_m$ are the ceramic surface temperature and metallic surface temperature, respectively. In Eq. (37), $\overline{z}$ is the distance from a point on the spherical shell surface to the spherical center. We should note that this point is separated from the central shell surface by a distance $-z$, which means that $\overline{z} = R+z$ and $R-h/2 \le \overline{z} \le R+h/2$.

In order to solve Eq. (37), we can represent the root as the follows (Bich and Tung, 2011):

$$T(\overline{z}) = T_m + \frac{\Delta T}{\int\limits_{R-h/2}^{R+h/2}\frac{d\overline{z}}{\overline{z}^2K(\overline{z})}}\int\limits_{R-h/2}^{\overline{z}}\frac{d\zeta}{\zeta^2K(\zeta)} \tag{38}$$

where, $\Delta T = T_c - T_m$ is the temperature gradient between the ceramic surface and metallic surface of the spherical shell. For simplicity, we just consider the linear distribution of the constituents in the spherical shell materials, i.e. $N = 1$ and:

$$K(\overline{z}) = K_m + K_{cm}\left[\frac{2(\overline{z}-R)+h}{2h}\right] \tag{39}$$

Introduction of Eq. (39) into Eq. (38) gives temperature distribution across the shell thickness as

$$T(z) = T_m + \frac{\Delta T}{I}\left\{\frac{4K_{cm}}{(K_c+K_m-2K_{cm}R_h)^2h}\left[\ln\frac{(K_c+K_m)h+2K_{cm}z}{2hK_m} - \ln\frac{2(R+z)}{2R-h}\right] + \frac{2(2z+h)}{(K_c+K_m-2K_{cm}R_h)(R+z)(2R-h)}\right\} \tag{40}$$

where $\overline{z}$ has been replaced by $z+R$ after integration.

We assume that the metallic surface temperature is kept constantly as the initial one. And, we substitute Eq. (40) to Eq. (8b) we get $\Phi_a = \Delta ThL/I$.

The explicit analytical expressions of $L, I$ are calculated and given in the Appendix A.

The solution is similar to the case written in (3.2.1), we then have found the form for the $q(\overline{W})$ of the spherical shell FGM in terms of the thickness Eq. (36), under the externally homogeneous pressure and the thermal conductance. Here, we replace $P$ by $L/I$ and $\Delta T = T_c - T_m$.

## 4. Numerical results and discussion

For an illustration, we consider the spherical shell FGM with such the constituents as aluminum (metal) and alumina (ceramic) with the well-known properties which has been used in Bich and Tung (2011), Duc and Cong (2013):

$$E_m = 70\ \text{GPa}, \alpha_m = 23\times10^{-6}\ {}^\circ\text{C}^{-1}, K_m = 204\ \text{W/mK}$$
$$E_c = 380\ \text{GPa}, \alpha_c = 7.4\times10^{-6}\ {}^\circ\text{C}^{-1}, K_c = 10.4\ \text{W/mK}$$

and, the Poisson's coefficient $\nu = 0.3$.

Effects of the elastic foundations on the nonlinear response of FGM shallow spherical shells are shown in Table 1 and Fig. 2. Obviously, elastic foundations played positive role on nonlinear static response of the FGM spherical shell: the large $K_1$ and $K_2$ coefficients are, the larger loading capacity of the shells is. It is clear that the elastic foundations can enhance the mechanical loading capacity for the FGM spherical shells, and the effect of Pasternak foundation $K_2$ on critical uniform external pressure is larger than the Winkler foundation$K_1$. For the FGM spherical shell without elastic foundations, in this case, the obtained results is identical to the result of Bich and Tung (2011).

Fig. 3 shows effects of curvature radius-to-thickness ratio $R/h$ on the nonlinear response of FGM shallow spherical shells subjected to external pressure. This figure shows that the effect of $R/h$ ratio (=70, 80 and 90) on the critical buckling pressure of shells is very strong, and the load bearing capability of the spherical shell is enhanced as $R/h$ ratio decreases.

Fig. 4 illustrates the effects of radius of base-to-curvature radius ratio $r_0/R$ (=0.4, 0.5 and 0.6) on the nonlinear response of FGM spherical shells under uniform external pressure. This figure shows that change of $r_0/R$ ratio is very sensitive with nonlinear response of the FGM spherical shells. In this figure, it is obviously to show that an effect of the ratio $r_0/R$ on a nonlinear static response of the shell is very unstable in postbuckling period.

Fig. 5 shows the effects of volume fraction index $N$ on the nonlinear axisymmetric static response of FGM spherical shells. As can be seen, the load-average deflection curves become lower when $N$ increases. However, the increase in the extremum-type buckling load and postbuckling load carrying capacity of the shell when $N$ reduces is presented by a bigger difference between upper and lower buckling loads. This conclusion also is reported by Bich and Tung (2011), Bich et al. (2012). Fig. 5 shows us that the elastic foundation enhances the loading ability of the spherical shell as the follows: the force acting on the spherical shell with the elastic foundation must be larger than the one acting on the FGM spherical shell with the inelastic foundation. Moreover, the additional elastic foundation reduces the snap-through significantly.

Fig. 6 analyzes the affects of in-plane restraint conditions and elastic foundations on the nonlinear response of clamped FGM shallow spherical shells with freely movable (FM) edges under uniform external pressure. In comparison with the FM case, the spherical shells with immovable clamped edges (IM) on elastic foundations have a comparatively higher capability of carrying

**Table 1**
Effect of elastic foundation on the nonlinear response of FGM shallow spherical shells ($R/h = 80$, $r_0/R = 0.3$, $N = 1$) with movable edges (FM).

| $W/h$ | 0.5 | | 1 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|
| | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0$ | $\mu = 0.1$ |
| ($K_1 = 0, K_2 = 0$)[a] | (0.0069) | (0.0064) | (0.0100) | (0.0092) | (0.0033) | (0.0048) | (0.0157) | (0.0195) |
| $K_1 = 100, K_2 = 0$ | 0.0089 | 0.0084 | 0.0140 | 0.0131 | 0.0192 | 0.0207 | 0.0356 | 0.0394 |
| $K_1 = 100, K_2 = 10$ | 0.0104 | 0.0099 | 0.0170 | 0.0161 | 0.0312 | 0.0327 | 0.0505 | 0.0544 |
| $K_1 = 50, K_2 = 20$ | 0.0109 | 0.0104 | 0.0180 | 0.0171 | 0.0352 | 0.0367 | 0.0555 | 0.0594 |

[a] The obtained results ( with $K_1 = K_2 = 0$ in brackets) are the same with Bich and Tung's one (2011, without elastic foundations).

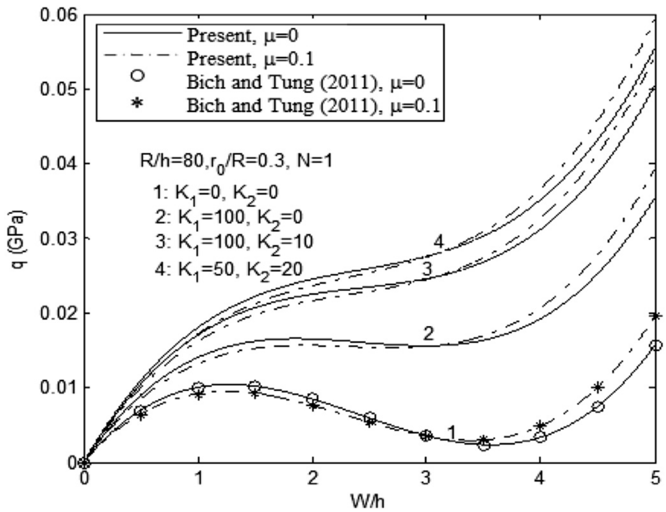


**Fig. 2.** Effects of the elastic foundations on the nonlinear response of FGM shallow spherical shells.

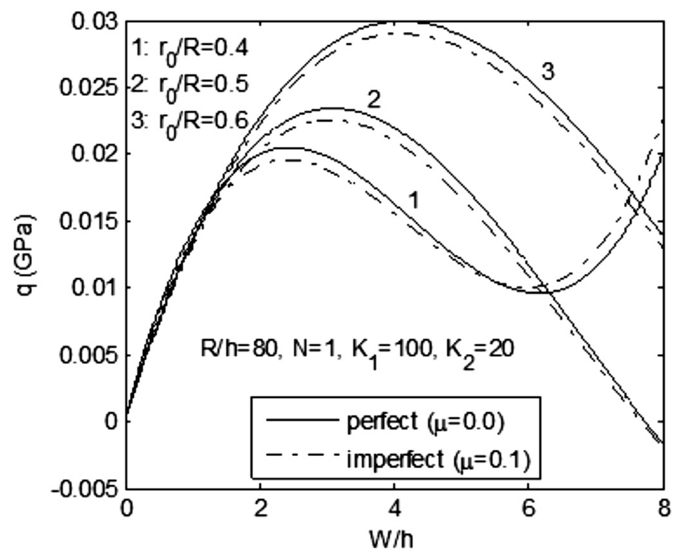


**Fig. 4.** Effects of $r_0/R$ on the stability of spherical shell FGM on the elastic foundation under an external pressure.

external pressure loads in a postbuckling period. However, the snap-through behavior of the FGM spherical shells with IM is very unstable. Strikingly, Fig. 6 shows that the useful effects of the elastic foundation (curves b) is more obvious than the inelastic one (curves a). Also, in the presence of the elastic foundation ($K_1 = 100, K_2 = 20$) the snap-through behavior of the FGM spherical shells is much more stable.

Table 2 and Fig. 7 present the effects of temperature and elastic foundations on the nonlinear response of FGM shallow spherical shells with clamped immovable edges (IM) under uniform external pressure. As shown in Fig. 7, the temperature makes the spherical shell to be deflected outward prior to mechanical loads acting on it. Under mechanical loads, outward deflection of the shell is reduced, and external pressure exceeds bifurcation point of load, an inward

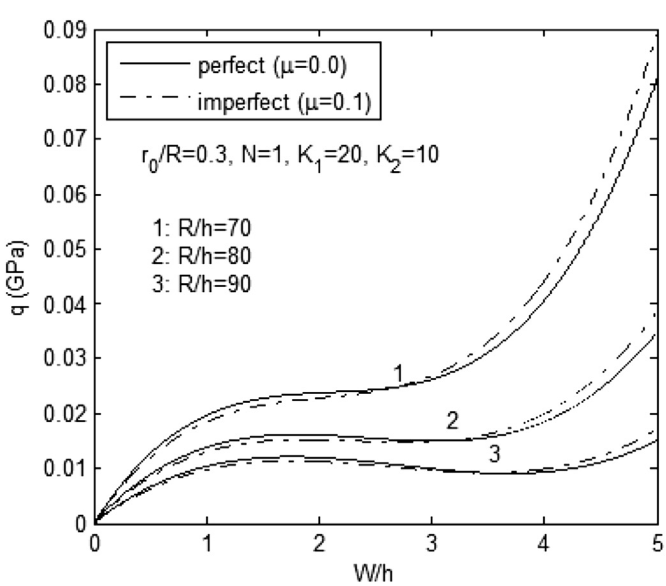


**Fig. 3.** Effects of $R/h$ on the stability of the spherical shell FGM on the elastic foundation under an external pressure.

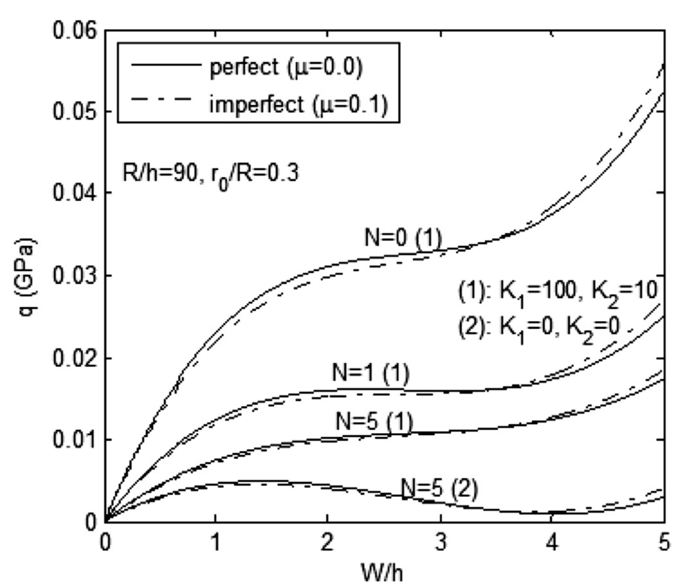


**Fig. 5.** Effects of index $N$ on the nonlinear response of FGM shallow spherical shells on elastic foundations.

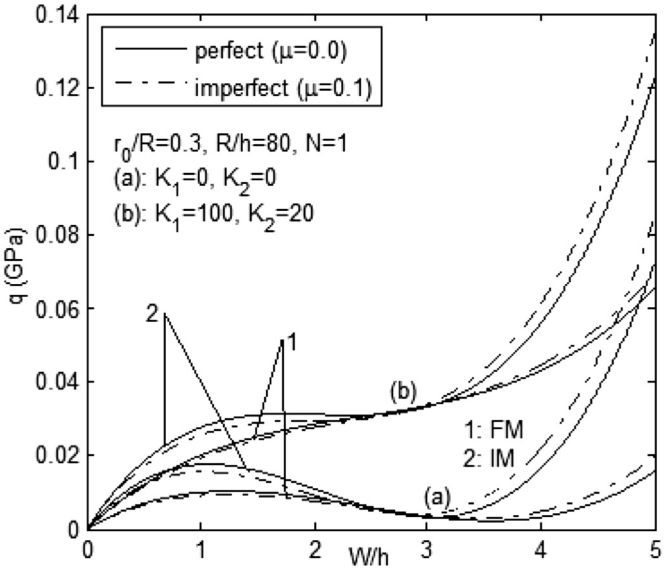


**Fig. 6.** Effects of in-plane restraint conditions and elastic foundations.

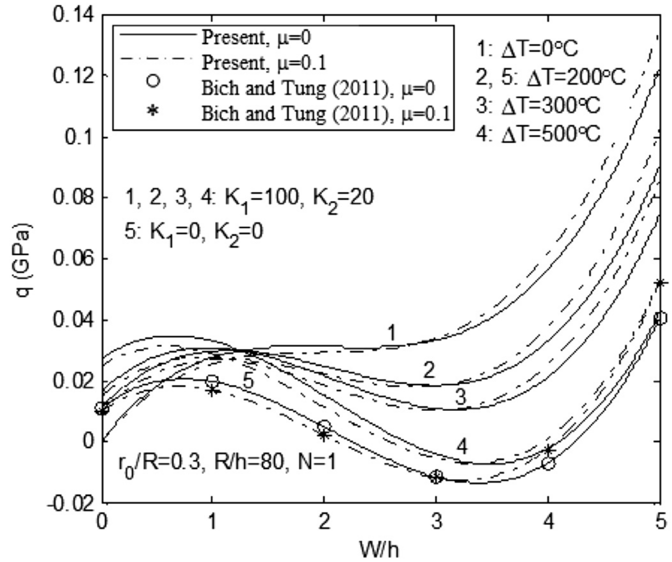


**Fig. 7.** Effects of uniform temperature rise and elastic foundations on the nonlinear response of FGM shallow spherical shells (IM).

deflection occurs. In this context, Fig. 7 also shows the bad effect of temperature on the nonlinear response of the FGM spherical shells. Indeed, the mechanical loading ability of the system has been reduced in the presence of temperature. In the absence of the elastic foundation (curve 5), there is a perfect agreement between our calculation and the well-known result reported by Bich and Tung (2011).

Fig. 8 depicts the interactive effect of FGM spherical shells on of temperature and imperfection on the thermomechanical response. This figure shows that the prefect spherical shells without temperature exhibit a more benign snap-through response and are more stable postbuckling behavior. This finding seems to be contradicting the regular behavior of the FGM plates in which the imperfect plates have loading capacity better than perfect one in postbuckling periods (Duc and Tung, 2011; Duc and Cong, 2013).

Interestingly, the effects of an elastic foundations has been presented in Fig. 9a and b. In these figures, we focus on the effects of imperfection on the nonlinear response of FGM shallow spherical shells (all FM edges) under uniform external pressure without elastic foundations (Fig. 9a) and resting on elastic foundations (Fig. 9b). This figures show that the effects of initial imperfection on the nonlinear response of the FGM spherical shells is significant. Obviously, an imperfection seems not very pronounced in postbuckling periods for the shell without elastic foundations. This result is consistent with those found by Bich and Tung (2011). The snap-through phenomenon in the absence of the elastic foundation is very strong. However, Fig. 9b shows the useful effects on the FGM spherical shells in the presence of the elastic foundation as the follows: the loading ability increases whereas the snap-through phenomenon has been reduced.

Fig. 10 investigates the effects of the pre-existent external pressure and the elastic foundation on the thermal loading ability of the IM spherical shells in the presence of temperature. The spherical shells behave and there is no snap-through phenomenon in the outward spherical shells as soon as the temperature change happens. Moreover, the effects of the imperfection is infinitesimal. Under the similar conditions, i.e. the same bending, the effects of the elastic foundation is very strong, i.e. the loading ability is much better, in other words, the buckling loads are much larger.

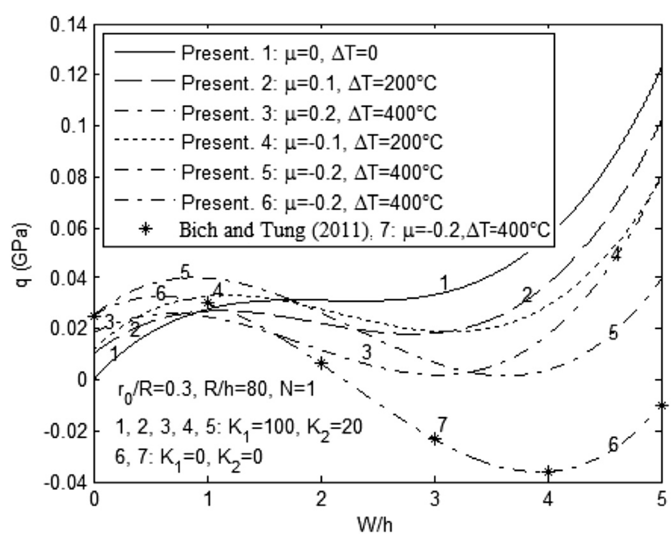


**Fig. 8.** Interactive effects of imperfection and temperature field on the nonlinear response of FGM shallow spherical shells (IM).

**Table 2**
Effect of temperature rise on the nonlinear response of FGM shallow spherical shells ($R/h = 80$, $r_0/R = 0.3$, $N = 1$, $K_1 = 100$, $K_2 = 20$) with immovable edges (IM).

| $W/h$ | 0.5 | | 1 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|
| | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0$ | $\mu = 0.1$ |
| $\Delta T = 0$ | 0.0181 | 0.0169 | 0.0275 | 0.0255 | 0.0562 | 0.0616 | 0.1224 | 0.1348 |
| $\Delta T = 200$ °C | 0.0354 | 0.0333 | 0.0405 | 0.0376 | 0.0435 | 0.0481 | 0.1012 | 0.1127 |
| $\Delta T = 300$ °C | 0.0441 | 0.0415 | 0.0470 | 0.0437 | 0.0372 | 0.0413 | 0.0906 | 0.1017 |
| $\Delta T = 500$ °C | 0.0613 | 0.0579 | 0.0600 | 0.0559 | 0.0245 | 0.0278 | 0.0694 | 0.0796 |

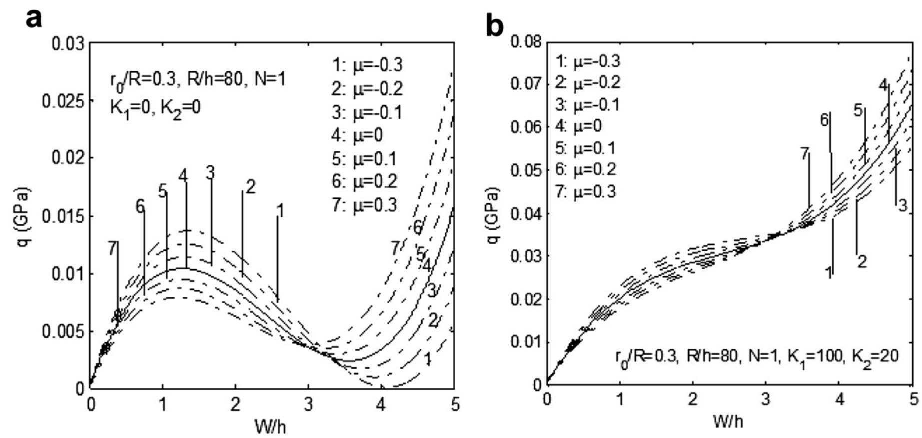


**Fig. 9.** a. Effects of imperfection on the nonlinear response of FGM shallow spherical shells without elastic foundations (FM). b. Effects of imperfection on the nonlinear response of FGM shallow spherical shells resting on elastic foundations (FM).

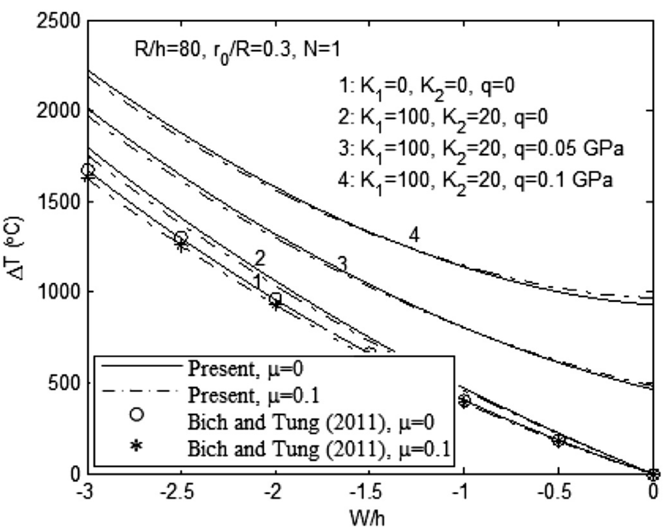


**Fig. 10.** Effects of pre-existent external pressure and elastic foundations on the thermal nonlinear response of FGM shallow spherical shells (IM).

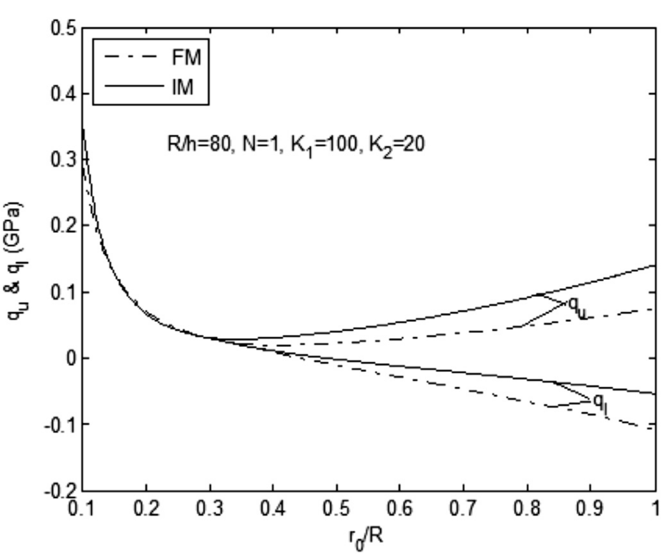


**Fig. 11.** Effects of the $r_0/R$ ratio on the upper ($q_u$) and lower ($q_l$) buckling loads of FGM shallow spherical shells.

**Table 3**
Effects of ratio $r_0/R$ and elastic foundations on range of upper and lower loads ($\Delta q$) for the perfect FGM spherical shell ($R/h = 80$, $N = 1$, $\mu = 0$).

| Case | Range $\Delta q$ | $r_0/R = 0.3$ | | $r_0/R = 0.4$ | | $r_0/R = 0.5$ | |
|---|---|---|---|---|---|---|---|
| | | $K_1 = 0$ $K_2 = 0$ | $K_1 = 100$ $K_2 = 5$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 100$ $K_2 = 5$ | $K_1 = 0$ $K_2 = 0$ | $K_1 = 100$ $K_2 = 5$ |
| FM | $\Delta q = q_u - q_l$ (GPa) | 0.008 | 0.001 | 0.0245 | 0.0187 | 0.0432 | 0.0392 |
| IM | $\Delta q = q_u - q_l$ (GPa) | 0.0143 | 0.0072 | 0.0306 | 0.0260 | 0.0492 | 0.0462 |

Fig. 11 presents the effects of the ratio $r_0/R$ on the upper and lower buckling loads for the perfect FGM spherical shells in both cases FM and IM. As can be observed, in small range of $r_0/R$, i.e. for very shallow shells, the upper and lower loads are almost identical and nonlinear response of the shell is predicted to be very mild. However, when ratio $r_0/R$ is higher, i.e. for deeper shells, intensity of snap-through (the difference between upper and lower loads) to be bigger.

Table 3 shows effects of ratio $r_0/R$ and elastic foundations on range of upper and lower loads ($\Delta q = q_u - q_l$) for the perfect FGM spherical shell ($R/h = 80$, $N = 1$, $\mu = 0$).

Table 3 shows us that the presence of an elastic foundation leads to a decrease of the intensity in both IM and FM cases. Whereas, the ratio $r_0/R$ increases with the intensity of snap-through.

Moreover, the intensity of snap-through in case of IM has been illustrated in Table 4. It is easy to show that the intensity of snap-through rises along with the increase of temperature.

Fig. 12 considers effects of temperature gradient on the nonlinear response of clamped immovable FGM shallow spherical shells (IM) under external pressure without elastic foundation – curves (a) and the FGM shell resting on elastic foundations – curves (b). It seems that bifurcation point are lower and the intensity of snap-through is weaker under temperature gradient in comparison with their uniform temperature rise (Fig. 7). This conclusion also is reported in Bich and Tung (2011). Interestingly, we should note that all curves of loads-deflections of the FGM spherical shell intersect at one point with different values of temperature change $\Delta T$. The understanding of this feature calls for a further investigation.

## 5. Conclusion

This paper considers the nonlinear axisymmetric response of FGM shallow spherical shells under uniform external pressure and temperature on elastic foundation using analytical approach. Two types of thermal condition are considered: The first type is assumed that the temperature is uniformly raised. The second type is that one value of the temperature is imposed on the upper

**Table 4**
Effects of temperature on range of upper and lower loads ($\Delta q$) for the perfect FGM spherical shell ($R/h = 80$, $r_0/R = 0.3$, $N = 1$, $\mu = 0$, $K_1 = 0$, $K_2 = 0$).

| | | $\Delta T = 0\ ^\circ\mathrm{C}$ | $\Delta T = 100\ ^\circ\mathrm{C}$ | $\Delta T = 200\ ^\circ\mathrm{C}$ | $\Delta T = 300\ ^\circ\mathrm{C}$ | $\Delta T = 500\ ^\circ\mathrm{C}$ |
|---|---|---|---|---|---|---|
| IM | $\Delta q = q_u - q_l$ (GPa) | 0.0143 | 0.0235 | 0.0342 | 0.0460 | 0.0730 |

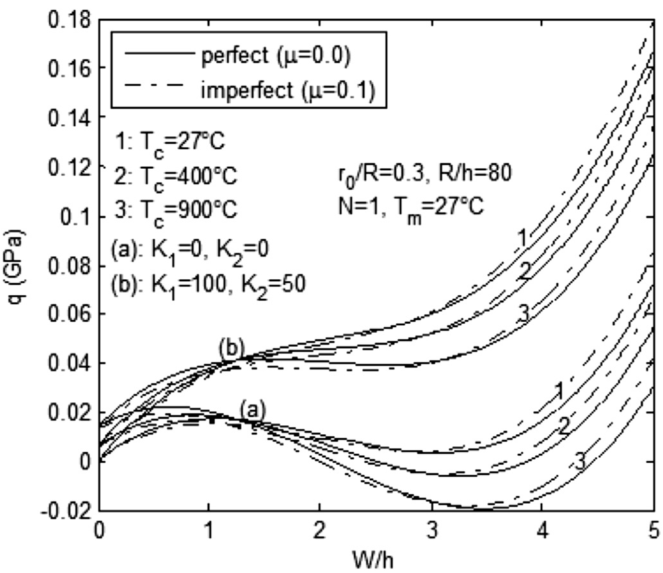


**Fig. 12.** Effects of the thermal conductance on the stable behavior of the spherical shell FGM under an external pressure (IM).

surface and the other value on the lower surface. The properties of materials are graded in thickness direction according to a power law function of thickness coordinate. Using classical shell theory, Bubnov–Galerkin method and stress function, obtained results show effects of external pressure, temperature, material and geometrical properties, imperfection, boundary conditions and particularly, the effects of elastic foundation on the nonlinear response of FGM shallow spherical shells. The snap-through behaviors of the FGM spherical shallow shells on the elastic foundations also are discussed carefully in this paper. Some results were compared with the ones of the other authors.

## Acknowledgments

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## Appendix A

$$E_1 = E_m h + \frac{E_{cm} h}{N+1}, E_2 = E_{cm} h^2 \left(\frac{1}{N+2} - \frac{1}{2N+2}\right)$$
$$E_3 = \frac{E_m h^3}{12} + E_{cm} h^3 \left(\frac{1}{N+3} - \frac{1}{N+2} + \frac{1}{4N+4}\right);$$
$$(\Phi_a, \Phi_b) = \int_{-h/2}^{h/2} (1, z) E(z) \alpha(z) \Delta T(z) dz$$

$$b_1^1 = \frac{64\overline{D}}{R_0^4 R_h^4} + \frac{3\overline{E_1}}{7R_h^2} + K_1 \frac{16\overline{D}}{21 R_0^4 R_h^4} + \frac{40\overline{D}}{7R_0^4 R_h^4} K_2; b_2^1 = \frac{976\overline{E_1}}{693 R_0^2 R_h^3};$$
$$b_3^1 = \frac{409\overline{E_1}}{693 R_h^3 R_0^2}; b_4^1 = \frac{848\overline{E_1}}{429 R_0^4 R_h^4}$$

$$R_h = R/h, R_0 = r_0/R, \overline{D} = D/h^3, \overline{E_1} = E_1/h, \overline{W} = W/h,$$
$$K_1 = \frac{k_1 r_0^4}{D}, K_2 = \frac{k_2 r_0^2}{D}$$

$$A = \frac{64\overline{D}}{R_0^4 R_h^4} + \frac{3\overline{E_1}}{7R_h^2} + K_1 \frac{16\overline{D}}{21 R_0^4 R_h^4} + \frac{40\overline{D}}{7R_0^4 R_h^4} K_2; B = \frac{1385\overline{E_1}}{693 R_0^2 R_h^3};$$
$$C = \frac{848\overline{E_1}}{143 R_0^4 R_h^4}$$

$$q_l = \frac{1}{3C^2}\left[B\left(3AC - 2B^2\right) - 2\left(B^2 - AC\right)^{3/2}\right]$$

$$q_u = \frac{1}{3C^2}\left[B\left(3AC - 2B^2\right) + 2\left(B^2 - AC\right)^{3/2}\right]$$

$$b_1^2 = \frac{64\overline{D}}{R_0^4 R_h^4} + \frac{(103-89\nu)\overline{E_1}}{126(1-\nu)R_h^2} + \frac{4\overline{E_2}}{(1-\nu)R_0^2 R_h^3} + K_1 \frac{16\overline{D}}{21 R_0^4 R_h^4} + \frac{40\overline{D}}{7R_0^4 R_h^4} K_2$$
$$b_2^2 = \left[\frac{(2637\nu-4331)\overline{E_1}}{2772(1-\nu)R_h^3 R_0^2}\right]; b_3^2 = \left[\frac{(42833-27103\nu)\overline{E_1}}{9009(1-\nu)R_0^4 R_h^4}\right];$$
$$b_4^2 = \left[\frac{(1526\nu-1746)\overline{E_1}}{693(1-\nu)R_0^2 R_h^3} - \frac{80\overline{E_2}}{7(1-\nu)R_0^4 R_h^4}\right]$$

$$P = E_m \alpha_m + \frac{E_m \alpha_{cm} + E_{cm} \alpha_m}{N+1} + \frac{E_{cm}\alpha_{cm}}{2N+1}, \overline{E_2} = \frac{E_2}{h^2}$$

$$I = \frac{4K_{cm}}{(K_c + K_m - 2K_{cm}R_h)^2} \ln \frac{K_c(2R_h - 1)}{K_m(2R_h + 1)} + \frac{8}{(K_c + K_m - 2K_{cm}R_h)\left(4R_h^2 - 1\right)}$$

$$\begin{aligned} L = & -\frac{K_{cm}\varsigma}{J^2}[\psi(R_h + 1/2) - 1] - \frac{\varsigma}{2J}\left(\psi - \frac{2}{2R_h - 1}\right) + \frac{\varsigma}{J^2}(K_m - K_c \\ & + \eta K_c) - \frac{K_{cm}\upsilon}{J^2}\left[R_h - \left(R_h^2 - 1/4\right)\psi\right] - \frac{\upsilon}{J}(1 - R_h\psi) \\ & - \frac{\upsilon}{2J^2 K_{cm}}\left(K_m^2 - K_c^2 + 2\eta K_m K_c\right) + \frac{K_{cm}E_{cm}\alpha_{cm}}{J^2}\left[\frac{1}{9} + \frac{4R_h^2}{3} \right. \\ & \left. - \left(\frac{1}{6} + \frac{4R_h^3}{3}\right)\psi\right] + \frac{2E_{cm}\alpha_{cm}}{J}\left[\frac{1}{6(2R_h - 1)} + R_h - R_h^2\psi\right] \\ & + \frac{E_{cm}\alpha_{cm}}{9J^2 K_{cm}^2}\left[4\left(K_m^3 - K_c^3\right) + 3K_c\left(K_c^2 + 3K_m^2\right)\eta\right] \end{aligned}$$

Where

$$\psi = \ln\frac{2R_h+1}{2R_h-1}; \eta = \ln\frac{K_c}{K_m}; J = K_c + K_m - 2K_{cm}R_h$$

$$\varsigma = (\alpha_c + \alpha_m)(E_c + E_m); \upsilon = E_{cm}(\alpha_c + \alpha_m) + \alpha_{cm}(E_c + E_m)$$

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