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ORIGINAL ARTICLE

The nonlinear stability of axisymmetric functionally graded material annular spherical shells under thermo-mechanical load

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ABSTRACT
The authors of this article investigate the nonlinear stability of axisymmetric functionally graded annular spherical shells with temperature-dependent material properties subjected to thermo-mechanical loads and resting on elastic foundations. Equilibrium and compatibility equations are derived by using the classical thin shell theory in terms of the shell deflection and the stress function. Approximate analytical solutions are assumed to satisfy simply supported boundary conditions and Galerkin method is applied to obtain the closed-form of load–deflection paths. An analysis is carried out to show the effects of material and geometrical properties and combination of loads on the stability of the shells.

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1. Introduction

Functionally graded materials (FGMs) consisting of metal and ceramic constituents have received increasing attention in structural applications. Smooth and continuous changes in material properties enable FGMs to avoid interface problems and unexpected thermal stress concentrations. By high performance heat resistance capacity, FGMs are now chosen for use as structural components exposed to severe temperature conditions, such as aircraft, aerospace structures, nuclear plants, and other engineering applications. This has prompted considerable researches to focus on the thermo-elastic, dynamic, and buckling analyses of FGMs in recent years.

The structures, as plates and shells, are usually supported by an elastic foundation. Therefore, it is necessary to account for effects of elastic foundations for a better understanding of the buckling and post-buckling behavior of the structures. By using the theory of elasticity and theory of shells, many approaches have been used to analyze the interaction between the structures and the elastic foundations. The problems of designing structures on elastic foundations were investigated by Gorbunov-Possadov et al. [1]. Regarding the nonlinear behavior of a spherical shell, Nie [2] studied nonlinear behavior of imperfect shallow spherical resting on Pasternak foundation by adopting the asymptotic iteration method with classical boundary conditions. The nonlinear buckling behavior of orthotropic truncated conical shells surrounded by an elastic medium was studied by Sofiyev [3] based on the finite deformation theory with von-Karman nonlinearity and Galerkin method to obtain the expressions for upper and lower critical axial loads. Duc et al. [4] investigated nonlinear axisymmetric response of FGM shallow spherical shells and double curved shallow FGM panels [5] on the Winkler–Pasternak foundations. The nonlinear post-buckling of imperfect eccentrically stiffened FGM double-curved thin shallow shell (with outside reinforced stiffeners) resting on elastic foundations and including temperature effects were also investigated by Duc and Quan [6] using the stress function and Galerkin procedure. Sofiyev [7] investigated buckling and the vibration analysis [8] of simply supported FGM truncated conical shells resting on the two-parameter elastic foundations. Nonlinear free vibration response, static, and the maximum transient response under uniformly distributed load of orthotropic thin spherical caps on elastic foundation have been obtained by Dumir [9].

Despite the evident importance in practical applications, to the best of our knowledge, there has been no publication on buckling FGM annular spherical shells. The annular spherical shell is not the same as the spherical shell, and the presence of the radii r_1, r_0 of the annular shell leads to the complicated calculation. After much time focused on researching, Anh et al. [10] recently announced the results of the buckling and post-buckling of thin annular spherical shells, with assuming the annular is un-axisymmetric shells. In this study, the authors wanted to develop research trying to apply the transformation, which was used by Ahn et al. [10], to solve the problems related to the FGM axisymmetric annular spherical shells under the thermo-mechanical loads and resting on Winkler–Pasternak-type elastic foundations. In addition, remarkably the material properties of the FGM annular spherical shells were changing and dependent on temperature, and the results were obtained with a very distinct difference between the bucking and post-buckling response of the axisymmetric and un-axisymmetric annular spherical shells, which is the reason for the authors to complete this study.

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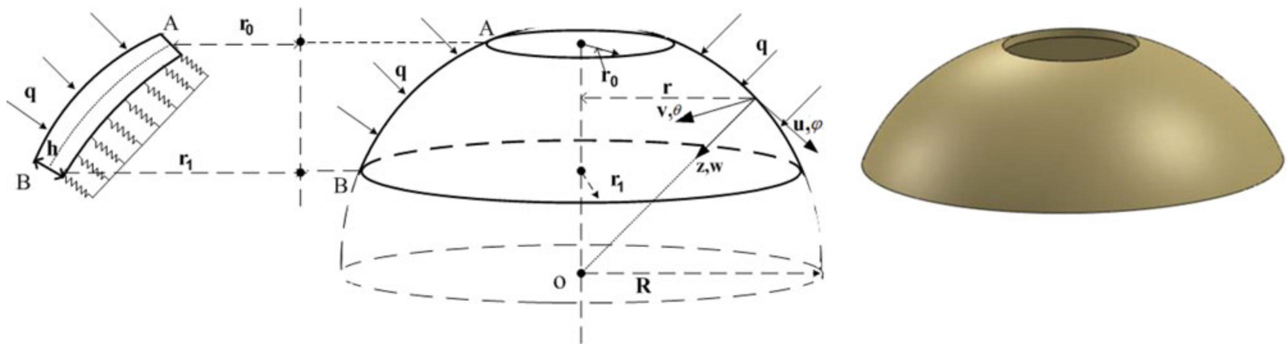

Figure 1. Configuration of an FGM axisymmetric annular spherical shell.

2. Governing equations

Consider that an axisymmetric annular spherical shell is subjected to external pressure q uniformly distributed on the outer surface as shown in Figure 1 where R is radius of curvature, h is thickness, and r_1, r_0 indicates the radii of annular.

In the present study, the classical shell theory is used to obtain the equilibrium and compatibility equations as well as expressions of buckling loads and nonlinear load–deflection curves of thin FGM axisymmetric annular spherical shells. It is appropriate to introduce a variable r, referred to as the radius of parallel circle with the base of shell and defined by $r = R\sin\varphi$. Moreover, due to shallowness of the shell it is approximately assumed that $\cos\varphi = 1$, $Rd\varphi = dr$.

2.1. Winkler–Pasternak-type elastic foundations

The FGM axisymmetric annular spherical shell is resting on the elastic foundation. For the elastic foundation, one assumes the two-parameter elastic foundation model proposed by Pasternak [4]. The foundation medium is assumed to be linear, homogenous, and isotropic. The bonding between the axisymmetric annular spherical shell and the foundation is perfect and frictionless. If the effects of damping and inertia force in the foundation are neglected, the foundation interface pressure may be expressed as: $q_e = k_1 w - k_2 \Delta w$, where $\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}$, w is the deflection of the annular spherical shell, k_1 is Winkler foundation modulus, and k_2 is the shear layer foundation stiffness of the Pasternak model.

2.2. Material properties of functionally graded shells

The axisymmetric annular spherical shell is made of FGM from a mixture of ceramics and metals, and is defined in a coordinate system (φ, θ, z), where φ and θ are in the meridional and circumferential direction of the shells, respectively, and z is perpendicular to the middle surface positive inwards.

Suppose that the material composition of the shell varies smoothly along the thickness by a simple power law in terms of the volume fractions of the constituents as:

$$V_c(z) = \left(\frac{2z+h}{2h}\right)^k, \; V_m(z) = 1 - V_c(z); \quad -\frac{h}{2} \le z \le \frac{h}{2}, \tag{1}$$

where k (volume fraction index) is a non-negative number that defines the material distribution.

Effective properties Pr_{eff} of the shell are determined by linear rule of mixture as:

$$\mathrm{Pr}_{eff}(z, T) = \mathrm{Pr}_m(T)V_m(z) + \mathrm{Pr}_c(T)V_c(z), \tag{2}$$

where Pr denotes a temperature-dependent material property, and subscripts m and c stand for the metal and ceramic constituents, respectively. Specific expressions of material coefficients are obtained by substituting Eq. (1) into Eq. (2) as:

$$\begin{aligned}
&[E(z,T),\ \upsilon(z,T),\ \rho(z,T),\ \alpha(z,T),\ K(z,T)] \\
&= [E_m(T),\ \nu_m(T),\ \rho_m(T),\ \alpha_m(T),\ K_m(T)] \\
&+ [E_{cm}(T),\ \nu_{cm}(T),\ \rho_{cm}(T),\ \alpha_{cm}(T),\ K_{cm}(T)]\left(\frac{2z+h}{2h}\right)^k,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
&E_{cm}(T) = E_c(T) - E_m(T),\ \nu_{cm}(T) \\
&\quad = \nu_c(T) - \nu_m(T),\ \rho_{cm}(T) = \rho_c(T) - \rho_m(T), \\
&\alpha_{cm}(T) = \alpha_c(T) - \alpha_m(T),\ K_{cm}(T) = K_c(T) - K_m(T).
\end{aligned} \tag{4}$$

It is evident from Eqs. (3) and (4) that the upper surface of the shell ($z = -h/2$) is ceramic-rich, while the lower surface ($z = h/2$) is metal-rich, and the percentage of ceramic constituent in the shell is enhanced when k increases. A material property Pr, such as the elastic modulus E, Poisson ratio ν, the mass density ρ, the thermal expansion coefficientα, and coefficient of thermal conduction K can be expressed as a nonlinear function of temperature [4, 5]:

$$\mathrm{Pr}(z, T) = P_0\left(P_{-1}T^{-1} + 1 + P_1T^1 + P_2T^2 + P_3T^3\right), \tag{5}$$

in which $T = T_0 + \Delta T(z)$, ΔT is temperature rise from stress-free initial state, and more generally, $\Delta T = \Delta T(z)$ and $T_0 = 300K$ (room temperature); P_0, P_{-1}, P_1, P_2, and P_3 are coefficients characterizing the constituent materials.

2.3. Fundamental relations and governing equations

According to the classical shell theory, the strains at the middle surface and the change of curvatures and twist are related to the displacement components u, υ, w for the axisymmetric annular spherical shell in the φ, θ, z coordinate directions, respectively, taking into account von Karman–Donnell nonlinear terms as [9, 10]:

$$\varepsilon_r^0 = \frac{\partial u}{\partial r} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \quad \varepsilon_\theta^0 = \frac{u}{r} - \frac{w}{R}; \quad \gamma_{r\theta}^0 = 0,$$

$$\chi_r = \frac{\partial^2 w}{\partial r^2}, \quad \chi_\theta = \frac{1}{r}\frac{\partial w}{\partial r}, \quad \chi_{r\theta} = 0, \tag{6}$$

where ε_r^0 and ε_θ^0 are the normal strains, $\gamma_{r\theta}^0$ is the shear strain at the middle surface of the shell, $\chi_r, \chi_\theta, \chi_{r\theta}$ are the changes of curvatures and twist.

The strains across the shell thickness at a distance z from the mid-plane are:

$$\varepsilon_r = \varepsilon_r^0 - z\chi_r, \quad \varepsilon_\theta = \varepsilon_\theta^0 - z\chi_\theta, \quad \gamma_{r\theta} = \gamma_{r\theta}^0 - z\chi_{r\theta}. \tag{7}$$

Using Eqs. (6) and (7), the geometrical compatibility equation of the shell is written as:

$$-\frac{1}{r}\frac{\partial \varepsilon_r^0}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_\theta^0}{\partial r}\right) = -\frac{\Delta w}{r} - \chi_r\chi_\theta. \tag{8}$$

By the Hooke law, the stress–strain relationships are defined as:

$$(\sigma_r, \sigma_\theta) = \frac{E(z)}{1-\nu^2}\left[(\varepsilon_r, \varepsilon_\theta) + \nu(\varepsilon_\theta, \varepsilon_r) - (1+\upsilon)\alpha\Delta T(1,1)\right], \quad \sigma_{r\theta} = 0, \tag{9}$$

where σ_r and σ_θ are the normal stress, $\sigma_{r\theta}$ is the shear stress at the middle surface of the shell in the spherical system coordinate, and ΔT denotes the increments of temperature from a surface to another one of the shells.

The force and moment resultants of the shell are expressed in terms of the stress components through the thickness as:

$$(N_{ij}, M_{ij}) = \int_{-h/2}^{h/2} \sigma_{ij}(1,z)dz, \, ij = (rr, \theta\theta). \tag{10}$$

In case of $(i = j = r)$ or $(i = j = \theta)$, for simplicity it is denoted as $N_{rr} = N_r, N_{\theta\theta} = N_\theta, M_{rr} = M_r, \ M_{\theta\theta} = M_\theta$.

By using Eqs. (7), (9), and (10) the constitutive relations can be given as:

$$(N_r, M_r) = \frac{(E_1, E_2)}{1-\nu^2}\left(\varepsilon_r^0 + \nu\varepsilon_\theta^0\right) - \frac{(E_2, E_3)}{1-\nu^2}(\chi_r + \nu\chi_\theta) - \frac{(\Phi_m, \Phi_b)}{1-\upsilon},$$

$$(N_\theta, M_\theta) = \frac{(E_1, E_2)}{1-\nu^2}\left(\varepsilon_\theta^0 + \nu\varepsilon_r^0\right) - \frac{(E_2, E_3)}{1-\nu^2}(\chi_\theta + \nu\chi_r) - \frac{(\Phi_m, \Phi_b)}{1-\upsilon},$$

$$(N_{r\theta}, M_{r\theta}) = 0. \tag{11}$$

From the relations one can write:

$$\varepsilon_r^0 = \frac{1}{E_1}(N_r - \nu N_\theta) + \frac{E_2}{E_1}\chi_r + \frac{\Phi_m}{E_1};$$

$$\varepsilon_\theta^0 = \frac{1}{E_1}(N_\theta - \nu N_r) + \frac{E_2}{E_1}\chi_\theta + \frac{\Phi_m}{E_1}; \ \gamma_{r\theta}^0 = 0, \tag{12}$$

$$M_r = \frac{E_2}{E_1}N_r - D_1(\chi_r + \nu\chi_\theta) - \frac{\Phi_b}{1-\upsilon};$$

$$M_\theta = \frac{E_2}{E_1}N_\theta - D_1(\chi_\theta + \nu\chi_r) - \frac{\Phi_b}{1-\upsilon}; \ M_{r\theta} = 0, \tag{13}$$

where

$$D_1 = \frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)}, \ E_1 = \int_{-h/2}^{h/2}\left[E_c + E_{cm}\left(\frac{2z+h}{h}\right)^k\right]dz = \left(E_m + \frac{E_{cm}}{k+1}\right)h,$$

$$E_2 = \int_{-h/2}^{h/2} z\left[E_c + E_{cm}\left(\frac{2z+h}{h}\right)^k\right]dz = h^2E_{cm}\left(\frac{1}{k+2} - \frac{1}{2k+2}\right),$$

$$E_3 = \int_{-h/2}^{h/2} z^2\left[E_c + E_{cm}\left(\frac{2z+h}{h}\right)^k\right]dz = \left(\frac{E_m}{12} + \frac{E_{cm}}{2(k+1)(k+2)(k+3)}\right)h^3,$$

$$(\Phi_m, \Phi_b) = \int_{-h/2}^{h/2}\left[E_m + E_{cm}\left(\frac{2z+h}{h}\right)^k\right] \times \left[\alpha_m + \alpha_{cm}\left(\frac{2z+h}{h}\right)^k\right]\Delta T(1,z)dz. \tag{14}$$

The nonlinear equilibrium equations of a perfect shallow spherical shell based on the classical shell theory [4] are as follows:

$$\frac{\partial N_r}{\partial r} + \frac{N_r}{r} - \frac{N_\theta}{r} = 0, \tag{15}$$

$$\frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r}\frac{\partial M_r}{\partial r} - \frac{1}{r}\frac{\partial M_\theta}{\partial r} + \frac{1}{R}(N_r + N_\theta) + \frac{1}{r}\frac{\partial}{\partial r}\left(rN_r\frac{\partial w}{\partial r}\right) + q - k_1w + k_2\Delta w = 0. \tag{16}$$

Equation (15) is identically satisfied by introducing a stress function F as:

$$N_r = \frac{1}{r}\frac{\partial F}{\partial r}, \ N_\theta = \frac{\partial^2 F}{\partial r^2}. \tag{17}$$

Substituting Eqs. (6), (12), and (17) into Eqs. (8) and substituting Eqs. (6), (13), and (14) into Eq. (16) leads to:

$$\frac{1}{E_1}\Delta\Delta F=-\frac{\Delta w}{r}-\frac{\partial^2 w}{\partial r^2}\frac{1}{r}\frac{\partial w}{\partial r}, \tag{18}$$

$$D_1\Delta\Delta w-\frac{\Delta F}{R}-\frac{1}{r}\frac{\partial F}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 F}{\partial r^2} = q-k_1w+k_2\Delta w. \tag{19}$$

Regularly, the stress function F should be determined by the substitution of deflection function w into compatibility equation (19) and solving the resulting equation. However, such a procedure is very complicated in mathematical treatment because the obtained equation is a variable coefficient partial differential equation. Accordingly, integration to obtain exact stress function $F(r,\theta)$ is extremely complex. Similarly, the problem of solving the equilibrium is in the same situation. Therefore, one should find a transformation to lead Eqs. (18) and (19) into constant coefficient differential equations. Suppose such a transformation:

$$w=w(\xi),\ F=F_0(\xi)e^{2\xi},\ \text{ where } r=r_0e^{\xi},\ \xi=\ln\frac{r}{r_0}. \tag{20}$$

From (20), it is easy to show that the shell is just a particular case of the annular shell in which $r_0=0$. Technically, the variable ξ unfortunately is not determined. Hence, we cannot apply this result for the spherical shell. The calculation of the annular shell using the transformation (20) is much more complicated. This, therefore, makes our calculation for FGM annular spherical shell in this article different from the obtained results in [4].

Substituting Eq. (20) into Eqs. (18) and (19) and establishing a lot of calculations lead to the transformed equations:

$$\frac{1}{E_1}\left(\frac{\partial^4 F_0}{\partial\xi^4}+4\frac{\partial^3 F_0}{\partial\xi^3}+4\frac{\partial^2 F_0}{\partial\xi^2}\right) = -\frac{r_0^2}{R}\frac{\partial^2 w}{\partial\xi^2}-\frac{1}{e^{2\varsigma}}\left(\frac{\partial^2 w}{\partial\xi^2}-\frac{\partial w}{\partial\xi}\right)\frac{\partial w}{\partial\xi}, \tag{21}$$

$$\begin{aligned}D_1&\left(\frac{\partial^4 w}{\partial\xi^4}-4\frac{\partial^3 w}{\partial\xi^3}+4\frac{\partial^2 w}{\partial\xi^2}\right)\\ &-\frac{r_0^2e^{4\xi}}{R}\left(\frac{\partial^2 F_0}{\partial\xi^2}+4\frac{\partial F_0}{\partial\xi}+4F_0\right)-\\ &-\left(\frac{\partial F_0}{\partial\xi}+2F_0\right)\left(\frac{\partial^2 w}{\partial\xi^2}-\frac{\partial w}{\partial\xi}\right)e^{2\xi}\\ &-\left(\frac{\partial^2 F_0}{\partial\xi^2}+2F_0+3\frac{\partial F_0}{\partial\xi}\right)\frac{\partial w}{\partial\xi}e^{2\xi}\\ &=qr_0^4e^{4\xi}-k_1r_0^4e^{4\xi}+k_2r_0^2e^{2\xi}.\end{aligned} \tag{22}$$

Equations (21) and (22) are the basic equations used to investigate the nonlinear buckling of FGM axisymmetric annular spherical shells.

3. Stability analysis

This section will investigate the nonlinear response of FGM axisymmetric annular spherical shell. The shell is assumed to be simply subjected to along the periphery and uniformly distributed load on the outer surface of the shell. Depending on the in-plane behavior at the edge of the boundary conditions will be considered in case the edges are simply supported and immovable. The boundary conditions are:

$$u=0,\ w=0,\ \frac{\partial^2 w}{\partial\xi^2}-\frac{\partial w}{\partial\xi}=0,\ \ N_r=N_0,\ \ N_{r\theta}=0,$$

$$\text{with } \xi=0 \text{ (i.e., } at\ r=r_0), \tag{23}$$

where N_0 is the normal force on the immovable edges.

To solve the equations of (21) and (22), we must find two dependent unknowns: $w(\varsigma)$ and $F_0(\varsigma)$. It can easily be seen that the unknowns form [7, 10, 11]:

$$w=We^{\xi}\sin(\beta\xi),\ \beta=\frac{m\pi}{a},\ a=\ln\frac{r_1}{r_0}, \tag{24}$$

used in [7, 10, 11], in which W is the maximum amplitude of deflection and m, n are the numbers of half waves in meridional and circumferential direction, respectively, satisfying the boundary conditions (23).

Substituting deflection function (24) into Eq. (21) and solving the obtained equation with boundary conditions (23) for unknown function F_0 leads to:

$$F_0=f_1e^{\xi}\sin(\beta\xi)+f_2e^{\xi}\cos(\beta\xi)+f_3\sin(2\beta\xi) +f_4\cos(2\beta\xi)+N_0\frac{r_0^2}{2}, \tag{25}$$

where

$$f_1=\frac{r_0^2E_1W}{R}\Delta_1,\quad f_2=\frac{r_0^2E_1W}{R}\Delta_2,$$

$$f_3=\frac{E_1W^2}{32\beta};\quad f_4=0,$$

$$\Delta_1=\frac{\left(-9-17\beta^2-7\beta^4+\beta^6\right)}{(9-22\beta^2+\beta^4)^2+(8\beta^3-24\beta)^2};$$

$$\Delta_2=\frac{6\left(\beta+2\beta^3+\beta^5\right)}{(9-22\beta^2+\beta^4)^2+(8\beta^3-24\beta)^2}. \tag{26}$$

Applying Galerkin method with the limits of integral is given by the formula:

$$\int_0^a \Phi e^{\xi}\sin(\beta\xi)d\xi=0, \tag{27}$$

where Φ is the left-hand side of Eq. (22) after substituting Eqs. (24) and (25), and we obtain the following equation:

$$q+N_0\overline{M_5}\,\overline{W}+N_0\overline{M_6}=\overline{M_1}\,\overline{W}+\overline{M_2}\,\overline{W^2}+\overline{M_3}\,\overline{W^3}, \tag{28}$$

where

$$\overline{M_1} = -\overline{D_1}\,\frac{(R_0^2 - R_1^2)R_h^4(25+\beta^2)\left(1+\beta^2\right)\beta}{4R_0\left[R_0^5 - R_1^5(-1)^m\right]} - \frac{\overline{E_1}(25+\beta^2)\,(\beta\Delta_1 + 3\Delta_2)\left(R_0^6 - R_1^6\right)R_h^2}{12\left[R_0^5 - R_1^5(-1)^m\right]R_0} - \frac{\overline{D_1}\left(R_0^6 - R_1^6\right)(25+\beta^2)\beta}{12R_0\,(9+\beta^2)\left[R_0^5 - R_1^5(-1)^m\right]}K_1 - \frac{\overline{D_1}\,R_h^2\left(R_0^4 - R_1^4\right)(25+\beta^2)\beta(3+\beta^2)}{8R_0\,(4+\beta^2)\left[R_0^5 - R_1^5(-1)^m\right]}K_2,$$

$$\overline{M_2} = \frac{\overline{E_1}\,R_h^3(25+\beta^2)}{R_0^2\,(625+250\beta^2+9\beta^4)} \times \left[\frac{15-37\beta^2}{16} - \beta^2(46+14\beta^2)\Delta_1 - \beta(3\beta^4 + 72\beta^2 + 165)\Delta_2\right],$$

$$\overline{M_3} = \frac{-\overline{E_1}\,R_h^4(25+\beta^2)\left(5+\beta^2\right)\beta\left(R_0^4 - R_1^4\right)}{256\,(4+\beta^2)\,R_0^3\left[R_0^5 - R_1^5(-1)^m\right]},$$

$$M_4 = \beta\,\frac{\left[r_0^5 - r_1^5(-1)^m\right]}{r_0\,(25+\beta^2)},$$

$$\overline{M_5} = \frac{\beta\left(\beta^2+3\right)\left(25+\beta^2\right)R_h^2\left(R_0^4 - R_1^4\right)}{8\,(4+\beta^2)\,R_0\left[R_0^5 - R_1^5(-1)^m\right]}, \qquad \overline{M_6} = 2R_h, \tag{29}$$

and putting

$$\overline{E_1} = \frac{E_1}{h}, \overline{E_2} = \frac{E_2}{h^2}, \overline{W} = \frac{W}{h}, R_h = \frac{h}{R}, R_0 = \frac{r_0}{R}, \overline{D_1} = \frac{D_1}{h^3}, N_0 = -ph, K_1 = \frac{h^4k_1}{D_1}, K_2 = \frac{h^2k_2}{D_1}.$$

Equation (28) is used to determine the buckling loads and nonlinear equilibrium paths of the FGM axisymmetric annular spherical shell under mechanical and thermal loads including the effect of elastic foundations.

3.1. Nonlinear mechanical stability analysis

The FGM axisymmetric annular spherical shell with the edges are simply supported and immovable. The shell is assumed to be only subjected to uniformly distributed load on the outer surface and rested on elastic foundations, meaning $N_0 = 0$.

Thus, Eq. (28) reduces to:

$$q = \overline{M_1}\,\overline{W} + \overline{M_2}\,\overline{W}^2 + \overline{M_3}\,\overline{W}^3. \tag{30}$$

The shells only exhibit extremum-type buckling when the material and geometric parameters satisfy specific conditions, i.e., loss of stability occurs at a limit point. The extremum pressure buckling load of the shell can be found from Eq. (30) with the condition $\overline{M_2}^2 - 3\,\overline{M_1}\,\overline{M_3} > 0$ and has the corresponding value:

$$q_{upper} = \frac{\overline{M_2} + \sqrt{\overline{M_2}^2 - 3\,\overline{M_1}\,\overline{M_3}}}{3\,\overline{M_3}} \times \left(\frac{5\,\overline{M_2}^2 + 6\,\overline{M_1}\,\overline{M_3} + 5\,\overline{M_2}\sqrt{\overline{M_2}^2 - 3\,\overline{M_1}\,\overline{M_3}}}{9\,\overline{M_3}}\right), \tag{31}$$

$$q_{lower} = \frac{\overline{M_2} - \sqrt{\overline{M_2}^2 - 3\,\overline{M_1}\,\overline{M_3}}}{3\,\overline{M_3}} \times \left(\frac{5\,\overline{M_2}^2 + 6\,\overline{M_1}\,\overline{M_3} - 5\,\overline{M_2}\sqrt{\overline{M_2}^2 - 3\,\overline{M_1}\,\overline{M_3}}}{9\,\overline{M_3}}\right). \tag{32}$$

The snap-through behavior may be predicted and intensity of the snap-through behavior is given by the difference between the value of upper and lower buckling load points. If the condition is not satisfied, there exists only one equilibrium shape and the deflection-load curve is stable.

3.2. Nonlinear thermo-mechanical stability analysis

A simply supported FGM axisymmetric annular spherical shell with immovable edges under external pressure and thermal load is considered. The condition expressing the immovability on the edges, $u = 0$ (on $r = r_0, r_1$), is fulfilled on the average sense as:

$$\int_0^{\pi}\int_{r_0}^{r_1}\frac{\partial u}{\partial r}r\,dr\,d\theta = 0 \Rightarrow \int_0^{\ln\frac{r_1}{r_0}}\left(\frac{\partial u}{\partial r}\right)_{\xi} r_0^2 e^{e^{\xi}}\,d\xi = 0. \tag{33}$$

From Eqs. (6), (11), and (17), one can obtain the following expression:

$$\frac{\partial u}{\partial r} = \frac{1}{E_1}\left(\frac{1}{r}\frac{\partial F}{\partial r} - \nu\frac{\partial^2 F}{\partial r^2}\right) + \frac{E_2}{E_1}\frac{\partial^2 w}{\partial r^2} + \frac{\Phi_m}{E_1} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \tag{34}$$

Substituting the transformation (20) into Eq. (34) leads to:

$$\left(\frac{\partial u}{\partial r}\right)_{\xi} = \frac{1}{E_1}\left[\frac{1}{r_0^2}\left(\frac{\partial F_0}{\partial \xi} + 2F_0\right) - \frac{\upsilon}{r_0^2}\left(\frac{\partial^2 F_0}{\partial \xi^2} + 3\frac{\partial F_0}{\partial \xi} + 2F_0\right)\right] + \frac{E_2}{E_1}\frac{1}{r_0^2 e^{2\xi}}\left(\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial w}{\partial \xi}\right) + \frac{w}{R} - \frac{1}{2r_0^2 e^{2\xi}}\left(\frac{\partial w}{\partial \xi}\right)^2 + \frac{\Phi_m}{E_1}. \tag{35}$$

Introducing Eqs. (24), (25), and (26) into Eq. (35) and then inserting the obtained equation into Eq. (33), performing some

Table 1. Material properties of the constituent materials of the considered FGM shell.

Material	Property	P_0	P_{-1}	P_1	P_2	P_3
Si_3N_4 (ceramic)	E (Pa)	348.43e9	0	−3.70e−4	2.160e−7	−8.946e−11
	ρ (kg/m^3	2370	0	0	0	0
	α (K^{-1})	5.8723e−6	0	9.095e−4	0	0
	k (W/mK)	13.723	0	0	0	0
	ν	0.24	0	0	0	0
SUS304 (metal)	E (Pa)	201.04e9	0	3.079e−4	−6.534e−7	0
	ρ (kg/m^3)	8166	0	0	0	0
	α (K^{-1})	12.330e−6	0	8.086e−4	0	0
	k (W/mK)	15.379	0	0	0	0
	ν	0.3177	0	0	0	0

calculations and arrangements we obtain:

$$N_0 = -\frac{\Phi_m}{(1-\nu)} + \frac{E_1}{8r_0^2(1-\nu)}\left(2\beta^2+\nu\right)W^2 + \frac{2E_1}{Rr_0(1-\nu)\left(r_0^2-r_1^2\right)} \left(\begin{array}{l}\left(r_0^3-(-1)^m r_1^3\right)\left(\nu\beta\Delta_1+(2\upsilon-1)\Delta_2+\dfrac{\beta}{(9+\beta^2)}\right)\\ -\dfrac{E_2}{E_1}R\beta\left(r_0-(-1)^m r_1\right)\end{array}\right)W, \tag{36}$$

which represents the normal stress on the immovable edge depending on thermal load and pre-buckling deflection.

Environment temperature is assumed to be uniformly raised from initial value T_0 at which the shell is thermal stress free, to the final one T_f and temperature change ($\Delta T = T_f - T_0$) is independent to thickness variable [12]. The temperature parameter Φ_m is expressed through ΔT by integration in (14) as:

$$\Phi_m = Ph\Delta T, \tag{37}$$

where $P = E_c\alpha_c + \dfrac{E_c\alpha_{mc} + E_{mc}\alpha_c}{k+1} + \dfrac{E_{mc}\alpha_{mc}}{2k+1}$.

Subsequently, employing this expression Φ_m in Eq. (36), and then substitution of the result N_0 into Eq. (28) leads to:

$$q = \frac{2PR_h}{(1-\nu)}\Delta T + \left(\overline{M_1} + \phi_1\Delta T - \phi_4\right)\overline{W} + \left(\overline{M_2} - \phi_2 - \phi_5\right)\overline{W}^2 + \left(\overline{M_3} - \phi_3\right)\overline{W}^3, \tag{38}$$

where the constants $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ are given in the Appendix.

From Eqs. (36) and (38), ΔT can be expressed formally as follows:

$$\Delta T = \frac{q}{\phi_1\overline{W} + \frac{2PR_h}{(1-\upsilon)}} - \frac{\left(\overline{M_1} - \phi_4\right)}{\phi_1\overline{W} + \frac{2PR_h}{(1-\upsilon)}}\overline{W} - \frac{\left(\overline{M_2} - \phi_2 - \phi_5\right)}{\phi_1\overline{W} + \frac{2PR_h}{(1-\upsilon)}}\overline{W}^2 - \frac{\left(\overline{M_3} - \phi_3\right)}{\phi_1\overline{W} + \frac{2PR_h}{(1-\upsilon)}}\overline{W}^3. \tag{39}$$

Equation (39) is used to determine the deflection-thermal load curve of the FGM axisymmetric annular spherical shell, with external pressure on the outer surface of the shell given. Because the properties of component materials are dependent on temperature, an iterative algorithm is used to obtain the nonlinear curve.

4. Results and discussion

Consider a FGM axisymmetric annular spherical shell made from Si_3N_4 (ceramic) and $SUS304$ (metal). The typical values of the coefficients of the FGM with temperature-dependent material properties mentioned in (2) are listed in Table 1 [4, 12].

In this section, the shell is assumed to be simply supported along boundary edges. In all figures, W/h denotes the dimensionless and maximum deflection of the shell.

First, the study investigated the behavior of the FGM axisymmetric annular spherical shell only subjected to uniformly distributed load; the effect of material constituent on the nonlinear response of the shell is shown in Figure 2. As can be seen, the load-deflection curve is lower when k increases. This is expected because the axisymmetric annular spherical shell is made from a mixture of ceramic and metal, but ceramic constituent has a much higher elastic modulus than the metal constituent and it is decreased when k increases. However, as observed in Figure 2, the enhancement in load carrying capacity of a ceramic-rich axisymmetric annular spherical shell is more severe because

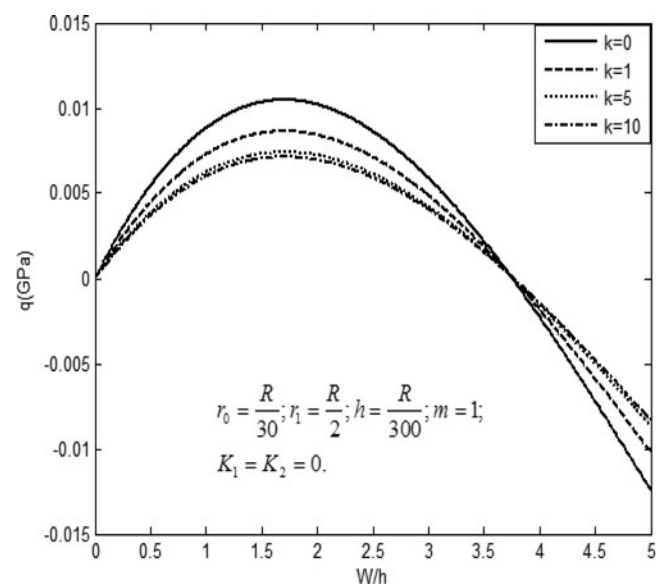

Figure 2. Influences of material constituents on the nonlinear response of the FGM axisymmetric annular spherical shell under uniformly distributed load.

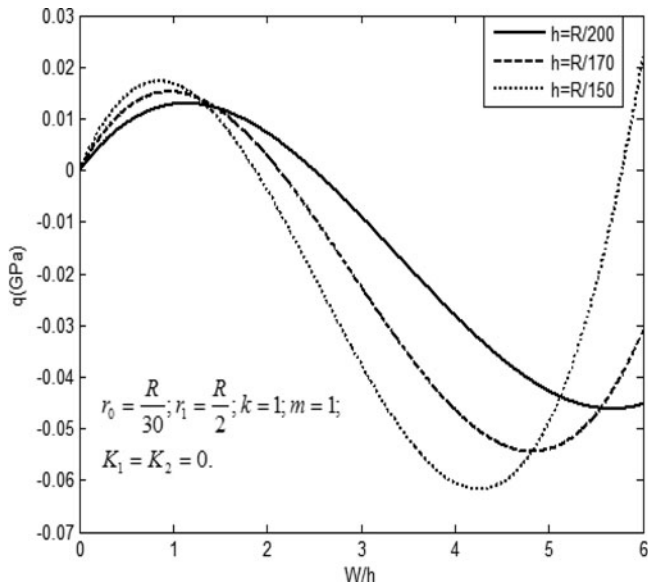

Figure 3. Influences of thickness ratio on the behavior of the FGM axisymmetric annular spherical shell under uniformly distributed load.

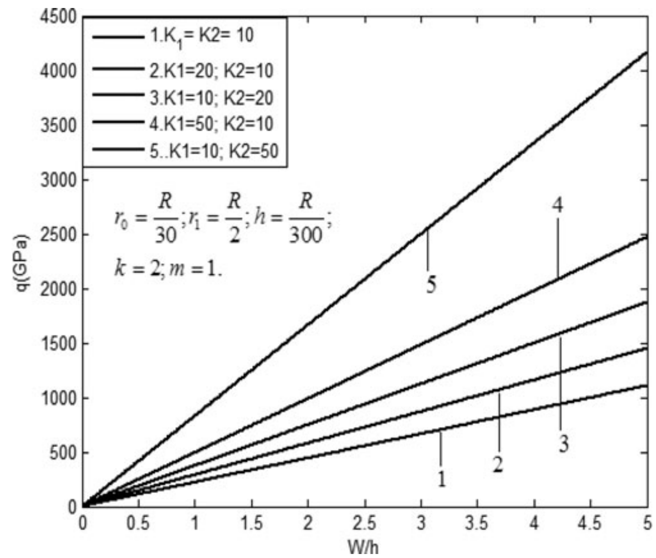

Figure 5. Impact of each elastic foundation on the nonlinear response of the FGM axisymmetric annular spherical shell.

after the load-deflection curve reached the extreme load conditions, it dropped rapidly on a steep slope, i.e., there is a significant difference between the upper and lower buckling state.

Geometrical parameters also have a great influence on the nonlinear response of FGM axisymmetric annular spherical shell. Specifically, the effects of thickness-radius ratio are shown in Figure 3. As can be observed, the R/h ratio decreases corresponding to the thicker axisymmetric annular spherical shell and the load carrying capacity is increased. But, the decrease in R/h ratio is accompanied by a drop of nonlinear equilibrium paths and a more severe snap-through response.

The elastic foundations are added with the aim to enhance the load carrying capacity and stability of the FGM axisymmetric annular spherical shell. Figure 4 considers the effects of both elastic foundations on the behavior of the shell. Interestingly, as soon as the elastic foundations are added, the nonlinear deflection-load curves become nearly-linear path. The enhancement of the Winkler foundation modulus and stiffness of the Pasternak foundation make a significant improvement in load carrying capacity of the shell. Figure 5 illustrates the impact of each elastic foundation on the behavior of the shell. As can be seen, the Pasternak foundation has a larger impact than Winkler foundation on nonlinear response of the shell and this is corresponding to the higher post-buckling curves. It illustrates the accuracy of the proposed approach. By combining Winkler and Pasternak foundation, the capacity of load carrying is significantly increased. Especially, these figures also exhibit the response of the shell as being more stable.

Influences of temperature gradient on the load carrying capacity of the shell are considered in Figure 6. Clearly, deflection-load curves lower when temperature increases. This

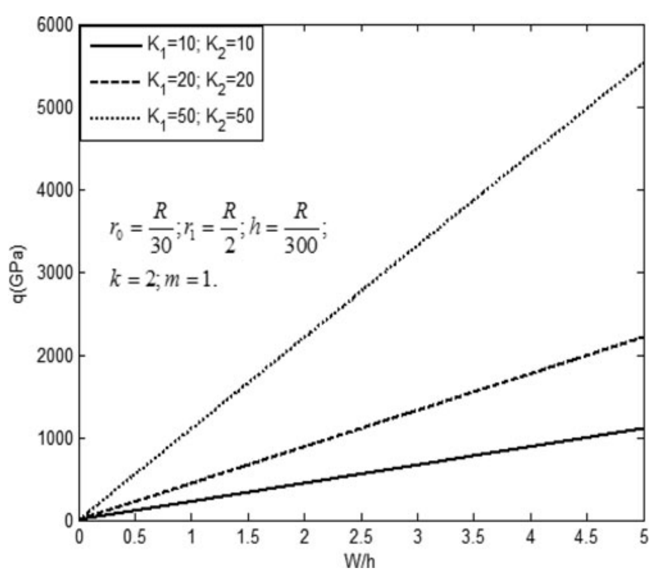

Figure 4. Effects of both elastic foundation on the nonlinear response of the FGM axisymmetric annular spherical shell.

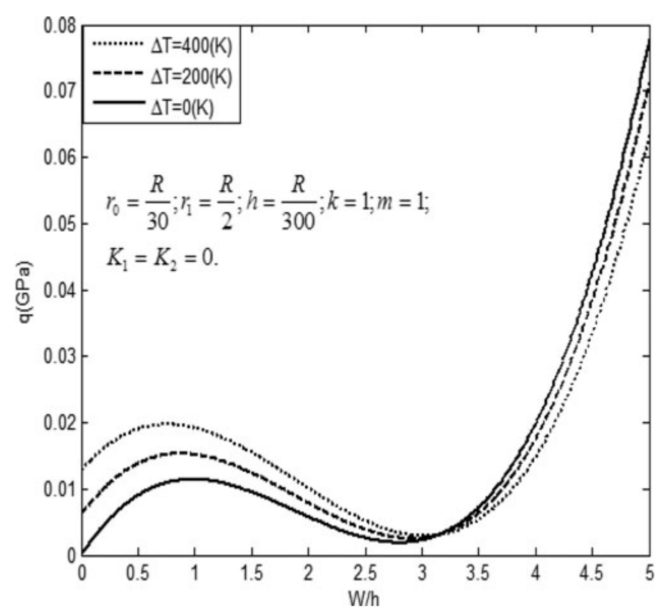

Figure 6. Effects of temperature change on capacity of load carrying of the FGM axisymmetric annular spherical shell.

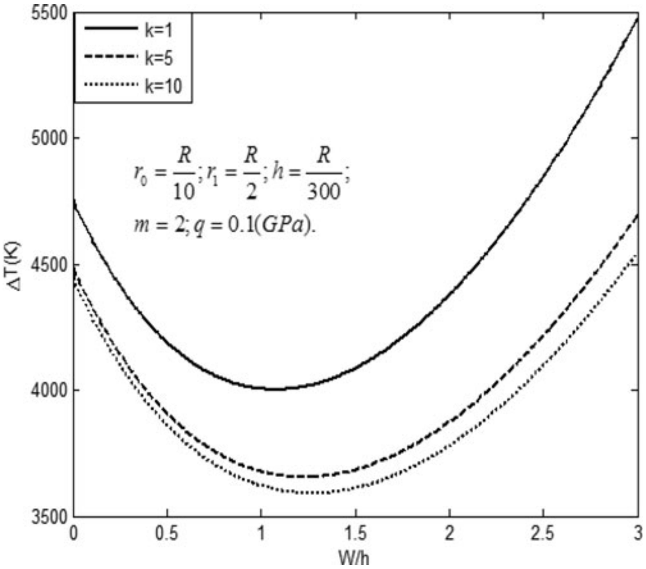

Figure 7. Influences of material constituents on the thermal load carrying capacity of FGM axisymmetric annular spherical shell.

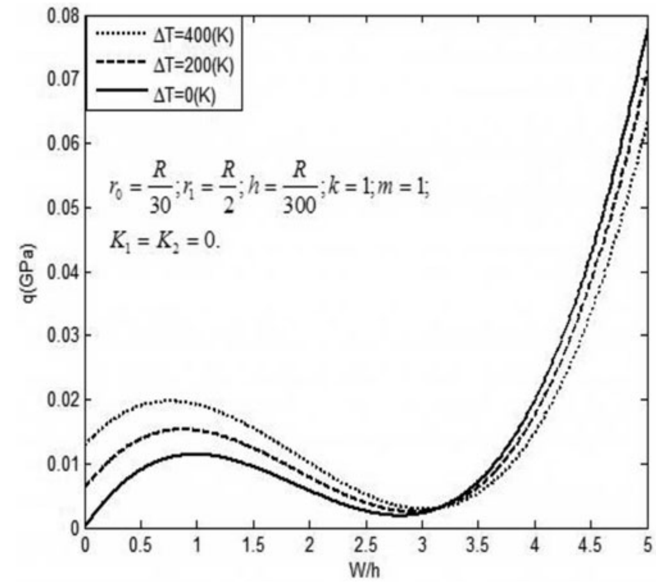

Figure 9. Effects of thickness-radius ratio on the thermal nonlinear response of the FGM axisymmetric annular spherical shell.

shows the bad effects of temperature on the nonlinear response of the shell. Indeed, the load carrying capacity has been reduced while temperature gradient increases. In Figure 6, temperature also makes the curves to be bifurcated load buckling and the shell was initially subjected to thermal stress in the presence of temperature. Figure 7 depicts the effects of the changing in material constituents affect the thermal loading ability. Obviously, the volume fraction index k increases, ceramic constituent ratio decreased, and deflection-load curves are significantly lower. Similar to the case of external load, ceramic's elastic modulus has a large impact on the behavior of the shell and it is expected.

Figure 8 considers interaction of both elastic foundations on the thermal load carrying capacity with pre-existent external pressure. As can be seen, load carrying ability is improved significantly when we increase the stiffness of the two elastic foundations. These values are not significantly increased but the deflection-thermal load curves are much higher, which showed that the shell is very sensitive to such changes. In addition, the results also show a large influence of the elastic foundations to the stability of the structure. This is to be expected and delivers the most effect in improving the durability of the structure.

Figures 9 and 10 investigate effects of geometrical parameters on the thermal nonlinear response and thermal load carrying capacity of the shell. Figure 9 illustrated the effects of curvature radius-to-thickness ratio R/h on the nonlinear response of the shells subjected to thermal load. Here, the curves are bifurcation buckling load type and there is no snap-through phenomenon in the outward shells as soon as the temperature change happens. Clearly, the thermal load bearing capability of the shell is enhanced as h/R ratio increases. Figure 10 depicts the effects of two base-curvature radius ratio r_1/r_0 on the nonlinear response of the shells. It is shown that the nonlinear response of the shells is very sensitive with change of r_1/r_0 ratio characterizing the

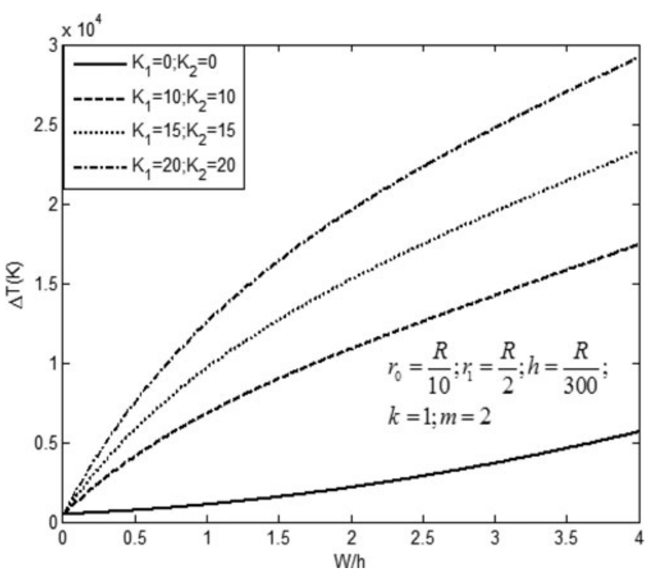

Figure 8. Effects of elastic foundations on the thermal loading ability of the FGM axisymmetric annular spherical shell with pre-existent external pressure.

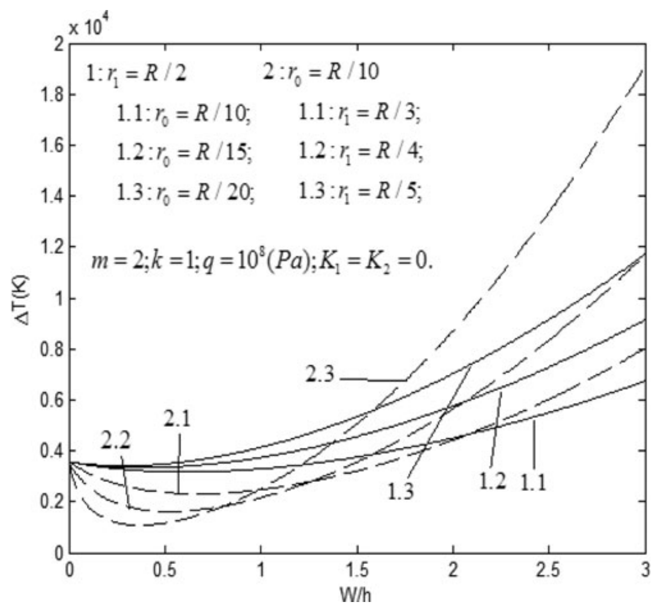

Figure 10. Effects of radii r_0 and r_1 on the thermal nonlinear response of the FGM axisymmetric annular spherical shell.

shallowness of the shell. Specifically, the enhancement of the upper buckling loads and the load carrying capacity in small range of deflection as r_1/r_0 increases is followed by very severe snap-through behaviors. In other words, in spite of possessing higher limit buckling loads, deeper shells exhibit a very unstable response from the post-buckling point of view. Furthermore, in the same effects of base-curvature radius ratio r_1/r_0, the load of the nonlinear response of the shells is higher when the shallowness of the shell (H) is smaller, where $H$is the distance between two radius r_1/r_0, and calculated is by:

$$H(r_1, r_0) = \sqrt{R^2 - r_0^2} - \sqrt{R^2 - r_1^2} = R\left[\sqrt{1 - \left(\frac{r_0}{R}\right)^2} - \sqrt{1 - \left(\frac{r_1}{R}\right)^2}\right].$$

The present figures of this study were obtained with a very distinct difference between the bucking and post-buckling response of the axisymmetric and un-axisymmetric annular spherical shells in Duc et al. [10].

5. Concluding remarks

The present article aims to propose an analytical approach to study the nonlinear buckling and post-buckling response of the thin axisymmetric FGM annular spherical shell with temperature-dependent material properties on elastic foundations subjected to mechanical and thermal loads. Based on the classical thin shell theory, the equilibrium and compatibility equations are derived in terms of the shell deflection and the stress function. This system of equations has been transformed into another system of more simple equations, so the appropriate formulas for FGM axisymmetric annular spherical shells are found to be a special case. Due to the properties of component materials being dependent on temperature, an iterative algorithm has been used to obtain the nonlinear curve. The effects of the material composition, volume fraction of constituent materials, Winkler and Pasternak type elastic foundations, and temperature on the nonlinear stability of the FGM shells are analyzed and discussed.

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Appendix

$$\phi_1 = \frac{P\Delta T}{8(1-\nu)} \frac{\beta\left(\beta^2+3\right)\left(\beta^2+25\right)\left(R_0^4 - R_1^4\right)R_h^2}{R_0\left(4+\beta^2\right)\left(R_0^5 - R_1^5(-1)^m\right)};$$

$$\phi_2 = \frac{2\,\overline{E_1}\,\beta\left(\beta^2+3\right)\left(25+\beta^2\right)\left(R_0^2+R_1^2\right)\left(R_0^3 - R_1^3(-1)^m\right)R_h^3}{8R_0^2\left(1-\upsilon\right)\left(4+\beta^2\right)\left(R_0^5 - R_1^5(-1)^m\right)} \times \begin{pmatrix} \nu\beta\Delta_1 + (2\nu-1)\,\Delta_2 + \\ + \dfrac{\beta}{(9+\beta^2)} \end{pmatrix} - \frac{2\,\overline{E_2}\,\beta^2\left(\beta^2+3\right)\left(\beta^2+25\right)\left(R_0^2+R_1^2\right)\left(R_0 - R_1(-1)^m\right)R_h^4}{8R_0^2\left(1-\upsilon\right)\left(4+\beta^2\right)\left(R_0^5 - R_1^5(-1)^m\right)};$$

$$\phi_3 = \overline{E_1}\left(2\beta^2+\upsilon\right)\frac{\beta\left(\beta^2+3\right)\left(25+\beta^2\right)\left(R_0^4 - R_1^4\right)R_h^4}{64\left(1-\upsilon\right)R_0^3\left(4+\beta^2\right)\left(R_0^5 - R_1^5(-1)^m\right)};$$

$$\phi_5 = \frac{\overline{E_1}\left(2\beta^2+\nu\right)R_h^3}{4R_0^2\left(1-\upsilon\right)}.$$

$$\phi_4 = \frac{4\,\overline{E_1}\left(R_0^3 - R_1^3(-1)^m\right)R_h^2}{R_0\left(1-\nu\right)\left(R_0^2 - R_1^2\right)} \times \left(\upsilon\beta\Delta_1 + (2\nu-1)\,\Delta_2 + \frac{\beta}{(9+\beta^2)}\right) - \frac{4\,\overline{E_2}\,\beta\left(R_0 - (-1)^m R_1\right)R_h^3}{R_0\left(1-\nu\right)\left(R_0^2 - R_1^2\right)}.$$