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The nonlinear dynamic and vibration of the S-FGM shallow spherical shells resting on an elastic foundations including temperature effects

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ABSTRACT

This study investigated the nonlinear dynamic and vibration of the S-FGM shallow spherical shells with ceramic-metal-ceramic layers (in two cases: non-axisymmetric and axisymmetric shells) on an elastic foundations (EF) with different types of boundary conditions in thermal environment. Material compositions of the shell are graded in the thickness direction according to a sigmoid law distribution in terms of the volume fractions of the constituents. The governing equations are derived by using the classical shell theory and Pasternak's two parameters EF. The motion equations of dynamic analysis are determined due to Galerkin method and the obtained equation is numerically solved by using Runge–Kutta method. The approximate solutions are assumed to satisfy the different types of boundary conditions. The criterion suggested by Budiansky–Roth is applied to determine the dynamic critical buckling load and the nonlinear dynamic response is found by numerical form. In numerical results, the effects of geometrical parameters, material properties, the EF, boundary conditions, mechanical loads and temperature on the nonlinear dynamic and vibration stability of the shells are investigated.

1. Introduction

Functionally graded materials (FGM) as a new class of advanced inhomogeneous composite materials have received considerable attention in many engineering applications for improved structural efficiency in space structures and nuclear reactors since they were first reported in Japan.

As a result, the problems relating to the thermo-elastic, dynamic, buckling analyses and vibration of structure made of FGMs have attracted attention of many researchers. For example, Houari et al. [1] investigated the thermo-elastic bending analysis of FGM sandwich plates using a new higher order shear and normal deformation theory by dividing the transverse displacement into bending, shear and thickness stretching part, the number of unknowns and governing equations for the present theory is reduced, significantly facilitating engineering analysis. Zidi et al. [2] by using a four variable refined plate theory, both a quadratic variation of the transverse shear strains across the thickness and the zero traction boundary conditions on the top and bottom surfaces of the plate are satisfied without using shear correction factors, to studied the bending analysis of FGM plates resting on elastic foundation and subjected to hygro-thermo-mechanical loading using a four variable refined plate theory. A simple and refined trigonometric higher-order beam theory is developed for bending and vibration of functionally graded beams has proposed by Bourada et al. [3] and the beauty of this theory is that, in addition to modeling the displacement field with only 3 unknowns as in Timoshenko beam theory, the thickness stretching effect is also included in the present theory. Jin et al. [4] presented a modified Fourier–Ritz approach for free vibration analysis of laminated FGM shallow shells with general boundary conditions in the framework of first-order shear deformation shallow shell theory with the displacement and rotation components of the shells are represented by the modified Fourier series consisted of standard Fourier cosine series and several closed-form auxiliary functions introduced to ensure and accelerate the convergence of the series representation. By using the Fourier-Ritz solution too, Yang et al. studied the vibration and damping analysis of sandwich plates with viscoelastic and FGM [5]. By Ashoori et al., in [6] the bifurcation-type buckling characteristics of heated FGM annular nanoplates resting on an elastic foundation and subjected to various types of thermal loading are carried out by presenting an exact analytical solution for the first time, or in [7] the nonlinear thermo-electrical stability of perfect/imperfect circular size-dependent FGM piezoelectric plates is studied according to modified couple stress theory, two types of thermal loading as well as two cases of boundary conditions are considered

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too in this present work. The paper [8] presents the nonlinear axisymmetric response of FGM shallow spherical shells with tangential edge constraints and resting on elastic foundations based on the first order shear deformation shell theory taking geometrical nonlinearity, initial geometrical imperfection by Tung HV. There are many more publication, however, this study focuses on The nonlinear dynamic and vibration so some research on nonlinear dynamics will be following overview.

For dynamic and vibration analysis of FGM structures, Deniz and Sofiyev [9] investigated the nonlinear dynamic buckling of FGM truncated conical shells subjected to axial compressive load varying as a linear function of time. Using the multiple scales method, Alijani et al. [10] derived nonlinear forced vibrations of FGM doubly curved shallow shells with a rectangular base, primary and subharmonic resonance responses of FGM shells are fully discussed. Chorfi and Houmat [11] investigated the nonlinear free vibration of a FGM doubly-curved shallow shell of elliptical plan-form using the p-version of the finite element method in conjunction with the blending function method. Ansari and Darvizeh [12] used a general analytical approach to investigate vibrational behavior of FGM shells, taking into account transverse shear deformation and rotary inertia effects. Ganapathi [13] published a result on the dynamic stability behavior of a clamped FGM spherical shell structural element subjected to external pressure load. Haddadpour et al. [14] researched free vibration analysis of simply supported FGM cylindrical shells for four sets of in-plane boundary conditions. Strozzi and Pellicano [15] analyzed the nonlinear vibrations of FGM circular cylindrical shells by using the Sanders–Koiter theory to model the nonlinear dynamics of the system in the case of finite amplitude of vibration. Sepiani et al. [16] researched the free vibration and buckling of a two-layered cylindrical shell made of inner FGM and outer isotropic elastic layer, subjected to combined static and periodic axial forces. Sofiyev [17] focused on the vibration and stability of freely supported FGM truncated and complete conical shells subjected to uniform lateral and hydrostatic pressures. Love's first approximation theory is used by Xiang et al. [18] to analyze the natural frequencies of rotating FGM cylindrical shells. Zhang et al. [19] presented an analysis on the nonlinear dynamics of a clamped–clamped FGM circular cylindrical shell subjected to an external excitation and uniform temperature change, based on the FSDT and Von Karman type non-linear strain–displacement relationship. Non-linear buckling analysis of FGM shallow spherical shells under pressure loads was presented by Ganapathi [20] by using finite element method, geometric non-linearity is assumed only on the meridional direction in strain-displacement relations. The nonlinear dynamic and vibration for the axisymmetric FGM shallow spherical shell was studied by Bich and Hoa in [21]. Bich et al. [22] studied the nonlinear static and dynamic buckling analysis of non-axisymmetric FGM shallow spherical shells with metal-ceramic layer including temperature effects using the approximated analytical method, geometric non-linearity is assumed in all strain-displacement relations, however the authors considered only an FGM shallow spherical shells subjected to external pressure loads varying as linear functions of time, $q = st$ (s – a loading speed).

The structures as plates and shells usually supported by an elastic foundation. By using the theory of elasticity and theory of shells, have many approaches to analyze the interaction between the structures and the EF, especially in the study of dynamic and vibration. For example, in [23] Duc investigated the nonlinear dynamic response and vibration of imperfect eccentrically stiffened thin FGM double curved shallow shells on elastic foundations. Used Reddy's higher order shear deformation shell theory, Duc et al. studied the nonlinear dynamic analysis of Sigmoid functionally graded circular cylindrical shells on EF in [24], in thermal environments [25] and with reinforced stiffeners in [26]. Duc et al. also considered nonlinear vibration and dynamic response of imperfect functionally graded thick double curved shallow shells resting on an EF and in thermal environments in [27] and the shells with piezoelectric actuators subjected to the combination of electrical, thermal, mechanical and damping loads in [28].

Duc et al. studied nonlinear thermo-mechanical dynamic analysis and vibration of higher order shear deformable piezoelectric functionally graded material sandwich plates in [29]. Hosseini et al. presented closed-from vibration analysis of thick annular functionally graded plates with integrated piezoelectric layers in [30].

Shah et al. [31] presented a study on the vibrations of FGM cylindrical shells based on the Winkler and Pasternak foundations. Besides also can find a lot of study on the nonlinear dynamic and vibration of structure resting on an EF in the Ref. [32] by Duc. Used Chebyshev series expansion, Nath and Alwar in [33] studied the nonlinear transient behavior of shallow spherical shells with and without damping effect. Bich et al. [34] considered nonlinear axisymmetric dynamic buckling and vibration of functionally graded shallow spherical shells under external pressure including temperature effects resting on an EF, however, the authors only considered the axisymmetric FGM shallow spherical shells with metal-ceramic layers.

The present paper aimed to propose an analytical approach to study the nonlinear dynamic and vibration of S-FGM spherical shallow shells (in two cases: non-axisymmetric and axisymmetric S-FGM shallow spherical shells) with ceramic-metal-ceramic layers and different types of boundary conditions resting on EF in thermal environment. Derivations of governing equations of these shells are based on the shell theory according to the Von Karman theory for moderately large deflection and small strain with the assumption of sigmoid law composition for the constituent materials. The approximate solutions are made to satisfy the different types of boundary conditions. The natural frequencies and nonlinear dynamic of the S-FGM spherical shells subjected to pressure loading are considered. The effects of boundary conditions, characteristics of functionally graded materials, temperatures and dimension ratios of the shells on their dynamical behaviors are investigated.

2. Governing equations

Consider a functionally graded (FGM) shallow spherical shell with metal – ceramic – metal layer, resting on an EF with radius of curvature R, base radius r_0 and thickness h in coordinate system (φ, θ, z), $-h/2 \le z \le h/2$ along the meridional, circumferential and radial-thickness directions respectively as shown in Fig. 1.

Suppose that the material composition of the shell varies smoothly along the thickness, applying a Sigmoid power law distribution for the shell with ceramic-metal-ceramic layers (S-FGM). The volume fractions of metal and ceramic V_m and V_c in the shell are assumed as [32].

$$V_m(z) = \begin{cases} (2z/h + 1)^k, \; -h/2 \le z \le 0 \\ (-2z/h + 1)^k, \; 0 \le z \le h/2 \end{cases}, \; V_c(z) = 1 - V_m(z). \tag{1}$$

where k (volume fraction index) is a non-negative number that defines the material distribution, subscripts m and c represent the metal and

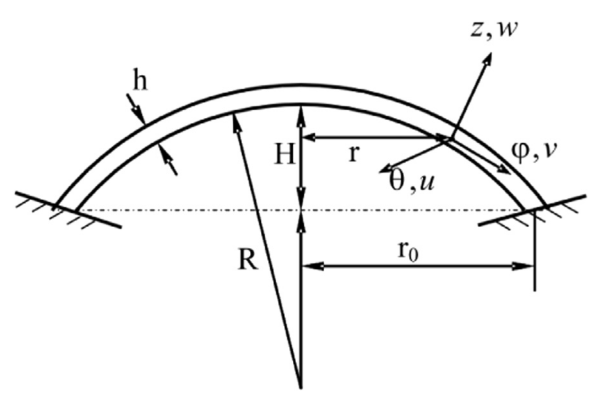

Fig. 1. Geometry of shallow spherical shell.

ceramic constituents, respectively. In this work, the material effective properties are obtained by substituting Eq. (1) and Poisson's ratio v is assumed to be constant.

As can be seen from Eq. (1), at $z = h/2$ and $-h/2$, the surfaces are fully ceramic and at $z = 0$, the surface is purely metal. Material properties corresponding to the isotropic shell with $k = 0$ and metal component will be increased as k increases as well.

The reaction-deflection relation of Pasternak foundation is given by [24,31,32].

$$q_e = k_1 w - k_2 \Delta w,\ \Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}, \tag{2}$$

where w is the deflection of the annular spherical shell, k_1 is Winkler foundation modulus and k_2 is the shear layer foundation stiffness of Pasternak model.

For the shell, the classical shell theory is used to obtain the equilibrium and compatibility equations as well as expressions of buckling loads and nonlinear load–deflection curves of thin S-FGM spherical shells. For a S-FGM spherical shell it is convenient to introduce a variable r, referred as the radius of parallel circle with the base of shell and defined by $r = R\sin\varphi$. Moreover, due to shallowness of the shell it is approximately assumed that $\cos\varphi = 1$, $Rd\varphi = dr$.

According to the classical shell theory, the strains at the middle surface and the change of curvatures and twist are related to the displacement components u, v, w in the φ, θ, z coordinate directions, respectively, taking into account Von Karman–Donnell nonlinear terms for non-axisymmetric shells, as [21,22]:

$$\begin{aligned}
\varepsilon_r^0 &= \frac{\partial u}{\partial r} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, & \chi_r &= \frac{\partial^2 w}{\partial r^2},\\
\varepsilon_\theta^0 &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{R} - \frac{w}{R} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2, & \chi_\theta &= \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2},\\
\gamma_{r\theta}^0 &= \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} & \chi_{r\theta} &= \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}.
\end{aligned} \tag{3}$$

where ε_r^0 and ε_θ^0 are the normal strains, γ_r^θ is the shear strain at the middle surface of the shell, $\chi_r, \chi_\theta, \chi_{r\theta}$ are the changes of curvatures and twist.

The strains across the shell thickness at a distance z from the mid-plane are:

$$\varepsilon_r = \varepsilon_r^0 - z\chi_r; \quad \varepsilon_\theta = \varepsilon_\theta^0 - z\chi_\theta; \quad \gamma_{r\theta} = \gamma_r^\theta - z\chi_{r\theta}. \tag{4}$$

Using Eqs. (3) and (4), the geometrical compatibility equation of the shell is written as

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_r^0}{\partial \theta^2} - \frac{1}{r}\frac{\partial \varepsilon_r^0}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varepsilon_\theta^0}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2}{\partial r \partial \theta}(r\gamma_r^0) = -\frac{\Delta w}{r} + \chi_{r\theta}^2 - \chi_r\chi_\theta, \tag{5}$$

The stress–strain relationships for the shell including temperature effect are defined by the Hooke's law [22,27,32].

$$\begin{aligned}
(\sigma_r,\ \sigma_\theta) &= \frac{E(z)}{1-\nu^2}[(\varepsilon_r,\ \varepsilon_\theta) + \nu(\varepsilon_\theta,\ \varepsilon_r) - (1+v)\alpha\Delta T(1,\ 1)],\\
\sigma_{r\theta} &= \frac{E(z)}{2(1+\nu)}\gamma_{r\theta}
\end{aligned} \tag{6}$$

where ΔT denotes the change of environment temperature from stress free initial state or temperature difference between the surfaces of the shell.

The force and moment resultants of the shell are expressed in terms of the stress components through the thickness as [32]:

$$(N_{ij},\ M_{ij}) = \int_{-h/2}^{h/2} \sigma_{ij}(1,\ z)dz,\ ij = (rr,\ \theta\theta,\ r\theta) \tag{7}$$

In case of $(i = j = r)$ or $(i = j = \theta)$ for simplicity denoted $N_{rr} = N_r$ $N_{\theta\theta} = N_\theta, M_{rr} = M_r, M_{\theta\theta} = M_\theta$.

By using Eqs. (4), (6), and (7) the constitutive relations can be given as

$$\begin{aligned}
(N_r,\ M_r) &= \frac{(E_1,\ E_2)}{1-\nu^2}(\varepsilon_r^0 + \nu\varepsilon_\theta^0) + \frac{(E_2,\ E_3)}{1-\nu^2}(\chi_r + \nu\chi_\theta) - \frac{(\Phi_m, \Phi_b)}{1-v},\\
(N_\theta,\ M_\theta) &= \frac{(E_1,\ E_2)}{1-\nu^2}(\varepsilon_\theta^0 + \nu\varepsilon_r^0) + \frac{(E_2,\ E_3)}{1-\nu^2}(\chi_\theta + \nu\chi_r) - \frac{(\Phi_m, \Phi_b)}{1-v},\\
(N_{r\theta},\ M_{r\theta}) &= \frac{(E_1,\ E_2)}{2(1+\nu)}\gamma_{r\theta}^0 + \frac{(E_2,\ E_3)}{1+\nu}\chi_{r\theta}.
\end{aligned} \tag{8}$$

From the relations one can write

$$\begin{aligned}
\varepsilon_r^0 &= \frac{1}{E_1}(N_r - \nu N_\theta) + \frac{E_2}{E_1}\chi_r + \frac{\Phi_m}{E_1},\\
\varepsilon_\theta^0 &= \frac{1}{E_1}(N_\theta - \nu N_r) + \frac{E_2}{E_1}\chi_\theta + \frac{\Phi_m}{E_1},\\
\gamma_{r\theta}^0 &= \frac{2(1+\nu)}{E_1}N_{r\theta} + \frac{2E_2}{E_1}\chi_{r\theta},
\end{aligned} \tag{9}$$

$$\begin{aligned}
M_r &= \frac{E_2}{E_1}N_r - D(\chi_r + \nu\chi_\theta) - \frac{\Phi_b}{1-v},\\
M_\theta &= \frac{E_2}{E_1}N_\theta - D(\chi_\theta + \nu\chi_r) - \frac{\Phi_b}{1-v},\\
M_{r\theta} &= \frac{E_2}{E_1}N_{r\theta} - D(1-\nu)\chi_{r\theta}.
\end{aligned} \tag{10}$$

By the classical shell theory and suppose that the shallow spherical shell is acted on by external uniform pressure q. According to Love's theory the equations of motion, in the general cases, are [21,22,32]

$$\begin{aligned}
&\frac{1}{r}\frac{\partial(rN_r)}{\partial r} + \frac{1}{r}\frac{\partial(N_{r\theta})}{\partial \theta} - \frac{N_\theta}{r} = \rho_1\frac{\partial^2 u}{\partial t^2}\\
&\frac{1}{r}\frac{\partial(rN_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial(N_\theta)}{\partial \theta} + \frac{N_{r\theta}}{r} = \rho_1\frac{\partial^2 v}{\partial t^2}\\
&\frac{1}{r}\left[\frac{\partial^2}{\partial r^2}(rM_r) + 2\left(\frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}\right) + \frac{1}{r}\frac{\partial^2 M_\theta}{\partial \theta^2} - \frac{\partial M_\theta}{\partial r}\right] + \frac{1}{R}(N_r + N_\theta)\\
&+\frac{\partial}{r\partial r}\left(rN_r\frac{\partial w}{\partial r} + N_{r\theta}\frac{\partial w}{\partial \theta}\right) + \frac{\partial}{r\partial \theta}\left(N_{r\theta}\frac{\partial w}{\partial r} + N_\theta\frac{\partial w}{r\partial \theta}\right) + q - k_1 w + k_2\Delta w = \rho_1\frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{11}$$

By taking the inertia forces $\rho_1\frac{\partial^2 u}{\partial t^2} \to 0$ and $\rho_1\frac{\partial^2 v}{\partial t^2} \to 0$ into consideration because of $u \ll w,\ v \ll w$, the first two equations of system (11) are satisfied by introducing the stress function f

$$N_r = \frac{\partial f}{r\partial r} + \frac{\partial^2 f}{r^2\partial \theta^2},\ N_\theta = \frac{\partial^2 f}{\partial r^2},\ N_{r\theta} = \frac{\partial f}{r^2\partial \theta} - \frac{\partial^2 f}{r\partial r\partial \theta}. \tag{12}$$

Substitution Eqs. (3), (9) and (12) into system of Eq. (5) and substituting Eqs. (3), (10) and (12) into Eq. (14) in terms of the stress function f and the deflection w lead to

$$\begin{aligned}
&\frac{1}{E_1}\Delta\Delta f = -\frac{\Delta w}{R} + \left(\frac{1}{r^2}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta}\right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{\partial^2 w}{r^2\partial \theta^2} + \frac{\partial w}{r\partial r}\right),\\
&q - k_1 w + k_2\Delta w = \rho_1\frac{\partial^2 w}{\partial t^2} + \frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)}\Delta\Delta w - \frac{1}{R}\Delta f - \left(\frac{\partial f}{r\partial r} + \frac{\partial^2 f}{r^2\partial \theta^2}\right)\frac{\partial^2 w}{\partial r^2}\\
&-\frac{1}{r^2}\frac{\partial^2 f}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2} - 2\left(\frac{\partial f}{r^3\partial \theta} - \frac{\partial^2 f}{r^2\partial r\partial \theta}\right)\frac{\partial^2 w}{\partial r\partial \theta} - \frac{\partial^2 f}{r\partial r^2}\frac{\partial w}{\partial r} + \frac{2}{r^2}\left(\frac{\partial f}{r^2\partial \theta} - \frac{\partial^2 f}{r\partial r\partial \theta}\right)\frac{\partial w}{\partial \theta}
\end{aligned} \tag{13}$$

in here

$$E_1 = E_c h + \frac{E_{mc}h}{(k+1)},\ E_2 = 0,$$

$$E_3 = \frac{E_c h^3}{12} + \frac{E_{mc}h^3}{2(k+1)(k+2)(k+3)},\ \rho_1 = \rho_c h + \frac{\rho_{mc}h}{(k+1)},$$

$$(\Phi_m,\ \Phi_b) = \int_{-h/2}^{h/2}\left[E_c + E_{cm}\left(\frac{2z+h}{h}\right)^k\right] \times \left[\alpha_c + \alpha_{cm}\left(\frac{2z+h}{h}\right)^k\right]\Delta T(1,\ z)dz,$$

In particular cases, for an axisymmetric shallow spherical shells, the partial derivative with respect to the variable θ will be equal to 0, the system of Eq. (13) leads to

$$\begin{aligned}
&\frac{1}{E_1}\Delta_s^2 f = -\frac{\Delta_s w}{R} - \frac{1}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial w}{\partial r},\\
&q - k_1 w + k_2\Delta_s w = \rho_1\frac{\partial^2 w}{\partial t^2} + \frac{E_3}{(1-\nu^2)}\Delta_s^2 w - \frac{1}{R}\Delta_s f - \frac{\partial f}{r\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{\partial w}{r\partial r}\frac{\partial^2 f}{\partial r^2}.
\end{aligned} \tag{14}$$

with $\Delta_s w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}$; $\quad \Delta_s f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}$

Eqs. (13) and (14) are the basic equations used to investigate the nonlinear dynamic and vibration stability of the S-FGM shallow spherical shells in general cases (non-axisymmetric shells) subjected to mechanical loads in thermal environment. These are nonlinear equations in terms of two dependent unknowns w and the stress function f.

3. Nonlinear dynamical analysis

3.1. Nonlinear dynamic of a S-FGM shallow spherical shells in general cases (non-axisymmetric shells)

The mentioned system of Eq. (13) combining with boundary conditions and initial conditions can be used in nonlinear dynamical analysis of FGM spherical shells. Suppose that the S-FGM spherical shell is clamped at its base edge $r = r_0$, the boundary conditions are

$$w = 0,\ \frac{\partial w}{\partial r} = 0,\ N_r = \frac{\partial f}{r\partial r} + \frac{\partial^2 f}{r^2\partial\theta^2} = 0 \text{ at } r = r_0. \tag{15}$$

The boundary conditions can be satisfied, if the deflection w and the stress function f, in general cases, are represented by

$$w = W\frac{r^2(r_0 - r)^2}{r_0^4}\sin(n\theta),\quad f = F\frac{r^2(r_0 - r)^2}{r_0^4}\sin(n\theta) \tag{16}$$

Applying Galerkin method to Eq. (13) in the range $0 \le \theta \le 2\pi,\ 0 \le r \le r_0$ yields a set of two equations with respect to W and F

$$F = \frac{E_1}{(6n^4 - 10n^2 + 19)}\left(\frac{W}{7R}r_0^2(2n^2 + 3) - \frac{2W^2 n}{7\pi}\right), \tag{17}$$

$$\frac{\partial^2 W}{6\partial t^2}\rho_1 + \frac{-7E_3(19 - 10n^2 + 6n^4)}{2(-1 + \nu^2)r_0^4}W + \frac{(3 + 2n^2)F}{2Rr_0^2} - \frac{2FWn}{\pi r_0^4} + \frac{Wk_1}{6} + \frac{2Wk_2}{r_0^2} = \frac{14q}{n\pi} \tag{18}$$

Eliminating F from two these equations leads to a nonlinear second-order ordinary differential equation for W

$$\frac{\partial^2 W}{6\partial t^2}\rho_1 + \left[\frac{-7E_3(19 - 10n^2 + 6n^4)}{2(-1 + \nu^2)r_0^4} + \frac{(3 + 2n^2)^2E_1}{14R^2(6n^4 - 10n^2 + 19)}\right]W + - \frac{3E_1(3 + 2n^2)}{(6n^4 - 10n^2 + 19)}\frac{nW^2}{7Rr_0^2\pi} + \frac{4n^2W^3E_1}{7r_0^4\pi^2(6n^4 - 10n^2 + 19)} + \frac{Wk_1}{6} + \frac{2Wk_2}{r_0^2} = \frac{14q}{n\pi} \tag{19}$$

where n is odd number.

The obtained Eq. (19) is a governing equation for dynamical analysis of S-FGM shallow spherical shells. Based on this equation the nonlinear vibration of S-FGM spherical shells can be investigated and the post-buckling analysis of shells can be performed.

3.2. In particular cases, for an axisymmetric S-FGM shallow spherical shells

In this section, the axisymmetric S-FGM spherical shells are assumed to be clamped along the periphery and subjected to external pressure uniformly distributed on the outer surface of the shells and depending on the in-plane behavior at the edge, two cases of boundary conditions, labeled Cases (1) and (2), will be considered.

Case (1). The edge is clamped and freely movable (FM) in the meridional direction. The associated boundary conditions are

$$\begin{aligned} &r = 0,\ \ \tfrac{\partial w}{\partial r} = 0,\ \ w = W,\\ &r = r_0,\ \ w = \tfrac{\partial w}{\partial r} = 0,\ \ N_r = 0. \end{aligned} \tag{20}$$

Case (2). The edge is clamped and immovable (IM). For this case, the boundary conditions are

$$\begin{aligned} &r = 0,\ \ \tfrac{\partial w}{\partial r} = 0,\ \ w = W,\\ &r = r_0,\ \ w = \tfrac{\partial w}{\partial r} = 0,\ \ N_r = N_{r0}. \end{aligned} \tag{21}$$

where N_{r0} is the fictitious compressive edge load rendering the edge immovable. With the consideration of the boundary conditions (20) and (21), the deflection w – in this case – is approximately assumed as follows

$$w = W\frac{(r_0^2 - r^2)^2}{r_0^4}. \tag{22}$$

Introduction of Eq. (22) into the first equation of Eq. (14) and integration of the resulting equation give stress function f with

$$\frac{\partial f}{\partial r} = -\frac{E_1W}{r_0^4R}\left(\frac{r^5}{6} - \frac{r_0^2r^3}{2}\right) - \frac{E_1W^2}{r_0^8}\left(\frac{r^7}{6} - \frac{2r_0^2r^5}{3} + r_0^4r^3\right) - \frac{E_1W}{3R}r + \frac{E_1W^2}{2r_0^2}r + N_{r0}r, \tag{23}$$

where $N_{r0} = 0$ for the spherical shells with movable clamped edge.

Substituting Eqs. (22) and (23) into the 2th equation of Eq. (14) and applying Galerkin method for the resulting equation yield

$$\frac{3\rho_1\partial^2W}{5\partial t^2} + \frac{6E_1W^3}{7r_0^4} - \frac{E_1W^2}{r_0^2R} + \left(\frac{64E_3}{r_0^4(1 - \nu^2)} + \frac{4E_1}{15R^2}\right)W + +\frac{4N_{r0}W}{r_0^2} - \frac{2N_{r0}}{R} + \frac{3k_1W}{5} + \frac{4k_2W}{r_0^2} = q \tag{24}$$

Eq. (24) is used to determine the dynamic responses of FGM shallow spherical shells under uniform external pressure without the effects of temperature conditions.

For the spherical shells with movable clamped edge, the governing equation for dynamical analysis of axisymmetric S-FGM shallow spherical shells is

$$\frac{3\rho_1\partial^2W}{5\partial t^2} + \frac{6E_1W^3}{7r_0^4} - \frac{E_1W^2}{r_0^2R} + \left(\frac{64E_3}{r_0^4(1 - \nu^2)} + \frac{4E_1}{15R^2} + \frac{3k_1}{5} + \frac{4k_2}{r_0^2}\right)W = q \tag{25}$$

A clamped S-FGM shallow spherical shell with immovable edge (that is, Case (2)) under simultaneous action of uniform external pressure q(in Pascals) and thermal load is considered. The condition expressing the immovability on the boundary edge, i.e. $u = 0$ on $r = r_0$, is fulfilled on the average sense as

$$\int_0^\pi\int_0^{r_0}\frac{\partial u}{\partial r}rdrd\theta = 0 \tag{26}$$

where

$$\frac{\partial u}{\partial r} = \frac{1}{E_1}\left(\frac{\partial f}{r\partial r} - \nu\frac{\partial^2 f}{\partial r^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{w}{R} + \frac{\Phi_m}{E_1} \tag{27}$$

Substituting Eqs. (22) and (23) into Eq. (27) and putting the result into Eq. (26) give

$$N_{r0} = -\frac{\Phi_m}{1 - \nu} + \left(\frac{(5\nu - 7)E_1}{36(1 - \nu)R} - \frac{2E_1}{(1 - \nu)r_0^2}\right)W + \frac{(35 - 13\nu)E_1}{72(1 - \nu)r_0^2}W^2, \tag{28}$$

which represents the compressive stress making the edge immovable. In what follows, specific expressions of thermo-mechanical load–deflection curves of axisymmetric S-FGM shallow spherical shells under uniform external pressure and thermal loads will be determined.

Environment temperature is assumed to be uniformly raised from initial value T_i at which the shell is thermal stress free, to final one T_f is independent to thickness variable. The thermal parameter can be expressed in terms $\Phi_m = Ph\Delta T$ with

$$P = E_c\alpha_c + \frac{E_c\alpha_{mc} + E_{mc}\alpha_c}{k + 1} + \frac{E_{mc}\alpha_{mc}}{2k + 1},\ \Delta T = T_f - T_i. \tag{29}$$

Subsequently, employing this expression Φ_m and then substitution of the result into Eq. (24) lead to

$$q = \frac{3\rho_1 \partial^2 W}{5\partial t^2} + \left(\frac{6}{7r_0^4} + \frac{4(35-13v)}{72(1-v)r_0^4}\right)E_1 W^3 + \left(\frac{24v-34}{13(1-v)R} - \frac{8}{(1-v)r_0^2}\right)\frac{E_1}{r_0^2}W^2 + \left(\frac{64E_3}{r_0^4(1-\nu^2)} - \frac{(5v-7)E_1}{18(1-v)R^2} + \frac{4E_1}{(1-v)Rr_0^2} + \frac{4E_1}{15R^2} + \frac{3k_1}{5} + \frac{4k_2}{r_0^2} - \frac{4\Phi_m}{r_0^2(1-v)}\right)W + \frac{2\Phi_m}{R(1-v)}. \tag{30}$$

The obtained Eqs. (25) and (30) is a governing equation for dynamical analysis of axisymmetric S-FGM shallow spherical shells. Based on this equation the nonlinear vibration of axisymmetric S-FGM spherical shells can be investigated and the post-buckling analysis of shells can be performed.

4. Nonlinear vibration analysis

4.1. In general cases, for S-FGM shallow spherical shells

Consider a S-FGM shallow spherical shell acted on by an uniformly distributed excited pressure load $q(t) = Qsin(\Omega t)$, the equation of motion has of the for

$$\frac{\partial^2 W}{6\partial t^2}\rho_1 + \left[\frac{7(E_2^2 - E_1E_3)}{2E_1(-1+\nu^2)r_0^4}(19 - 10n^2 + 6n^4) + \frac{(3+2n^2)^2 E_1}{14R^2(6n^4 - 10n^2 + 19)}\right]W + -\frac{3E_1(3+2n^2)}{(6n^4 - 10n^2 + 19)}\frac{nW^2}{7Rr_0^2\pi} + \frac{4n^2W^3E_1}{7r_0^4\pi^2(6n^4 - 10n^2 + 19)} + \frac{Wk_1}{6} + \frac{Wk_2}{r_0^2} = 14\frac{Qsin(\Omega t)}{n\pi} \tag{31}$$

From Eq. (31) the fundamental frequencies of natural vibration of the shell ω_n can be determined by the relation

$$\omega_n^2 = \frac{-21E_3(19 - 10n^2 + 6n^4)}{\rho_1(-1+\nu^2)r_0^4} + \frac{6E_1(3+2n^2)^2}{\rho_1 14R^2(6n^4 - 10n^2 + 19)} + \left(\frac{k_1}{\rho_1} + \frac{12k_2}{\rho_1 r_0^2}\right) \tag{32}$$

By putting

$$a_1 = \frac{-21E_3(19 - 10n^2 + 6n^4)}{\rho_1(-1+\nu^2)r_0^4} + \frac{6E_1(3+2n^2)^2}{14\rho_1 R^2(6n^4 - 10n^2 + 19)},$$
$$a_2 = -\frac{1}{\rho_1}\frac{18nE_1(3+2n^2)}{7Rr_0^2\pi(6n^4 - 10n^2 + 19)},$$
$$a_3 = \frac{1}{\rho_1}\frac{24n^2E_1}{7r_0^4\pi^2(6n^4 - 10n^2 + 19)},$$
$$H = \frac{84Q}{n\pi\rho_1}, \quad R_h = R/h,\ R_0 = r_0/R,\ \overline{D} = D/h^3,\ \overline{\rho}_1 = \rho_1 h, \tag{33}$$

$$K_1 = \frac{k_1 r_0^4}{D},\ K_2 = \frac{k_2 r_0^2}{D},\ b_1 = \frac{\overline{D}}{6\overline{\rho}_1 R_0^4 R_h^4},\ b_2 = \frac{12\overline{D}}{\overline{\rho}_1 R_0^4 R_h^4}.$$

The Eq. (31) can be rewritten as

$$\frac{\partial^2 W}{\partial t^2} + (a_1 + b_1K_1 + b_2K_2)W + a_2W^2 + a_3W^3 - H\sin(\Omega t) = 0 \tag{34}$$

and putting

$$\omega_n = \sqrt{(a_1 + b_1K_1 + b_2K_2)},$$

$$M = \frac{a_2}{(a_1 + b_1K_1 + b_2K_2)}, \quad N = \frac{a_3}{(a_1 + b_1K_1 + b_2K_2)}.$$

Eq. (34) can be rewritten in the form

$$\frac{\partial^2 W}{\partial t^2} + \omega_n^2(W + MW^2 + NW^3) - H\sin(\Omega t) = 0 \tag{35}$$

For seeking amplitude-frequency characteristics of nonlinear vibration, substitute $W = A\sin\Omega t$ into Eq. (34), the nonlinear Eq. (35) becomes

$$\omega_n^2(A\sin\Omega t + M(A\sin\Omega t)^2 + N(A\sin\Omega t)^3) - A\Omega^2\sin\Omega t - H\sin(\Omega t) = 0 \tag{36}$$

Multiply both sides of Eq. (36) by $\sin\Omega t$ and then integrate from 0 to $\pi/2\Omega$, and the amplitude-frequency relation of nonlinear forced vibration is obtained as

$$\frac{\Omega^2}{\omega_n^2} - 1 - \frac{8}{3}\frac{MA}{\pi} - \frac{3}{4}A^2N + \frac{H}{A\omega_n^2} = 0 \tag{37}$$

By denoting $\alpha^2 = \frac{\Omega^2}{\omega_n^2}$ as the frequency ratio, Eq. (37) becomes

$$\alpha^2 - \left(1 + \frac{8MA}{3\pi} + \frac{3}{4}A^2N\right) + \frac{H}{A\omega_n^2} = 0 \tag{38}$$

For the free nonlinear vibration, the frequency-amplitude relation is

$$\omega_{NL}^2 = \omega_n^2\left(1 - \frac{8}{3\pi}MA + \frac{3N}{4}A^2\right) - \frac{4H}{A}. \tag{39}$$

where ω_{NL} is the non-linear vibration frequency and A is the amplitude of non-linear vibration.

4.2. In particular cases, for axisymmetric S-FGM shallow spherical shells

The fundamental frequency of axisymmetric S-FGM shallow spherical shells with movable and immovable edge, respectively can be determined by

$$\omega_n = \sqrt{\frac{5}{3\rho_1}\left(\frac{64E_3}{r_0^4(1-\nu^2)} + \frac{4E_1}{15R^2} + \frac{3k_1}{5} + \frac{4k_2}{r_0^2}\right)} \tag{40a}$$

$$\omega_n = \sqrt{\frac{5}{3\rho_1}\left(\frac{64E_3}{r_0^4(1-\nu^2)} - \frac{(5v-7)E_1}{18(1-v)R^2} + \frac{4E_1}{(1-v)Rr_0^2} + \frac{4E_1}{15R^2} + \frac{3k_1}{5} + \frac{4k_2}{r_0^2} - \frac{4\Phi_m}{r_0^2(1-v)}\right)} \tag{40b}$$

Consider an axisymmetric S-FGM shallow spherical shells acted on by an uniformly external pressure $q = Q\sin\Omega t$, the nonlinear dynamic responses of shells can be obtained by solving Eqs. (25) and (30) combined with initial conditions to be assumed as $W(0) = 0$, $\frac{\partial W}{\partial r}(0) = 0$ by using the Runge - Kutta iteration schema.

For investigation the nonlinear dynamic buckling of the shells under linear-time loading, we need to seek the critical dynamic buckling loads. They can be evaluated based in the displacement responses obtained from the motion Eqs. (25) and (30) with $q = ct$ (c is a loading speed) and the Budiansky-Roth criterion is employed here as it is widely accepted in [13,14].

5. Numerical results and discussion

5.1. Comparison study

To validate proposed approach, consider a clamped FGM shallow spherical shell subjected to uniform external pressure varying as linear functions of time, $q = st$ (s – a loading speed) in the absence of elastic foundations.

In this discussion, the clamped FGM spherical shell is made of silicon nitride (Si_3N_4) and steel (SUS 304). The Young's modulus is $E_c = 348.43$ GPa, $E_m = 201.04$ GPa respectively. Critical dynamic buckling of a clamped FGM shallow spherical shell is calculated by using explicit expression (19) with $k_1 = k_2 = 0$ and compared in Tables 1a and 1b with those reported by Bich et al. in [14] utilizing the well-known criterion suggested by Budiansky and Roth with the same geometrical parameters.

The comparison results show a good fit between present results and the results in [14], and shows the reliability of the calculation method and obtained results.

Table 1a
Comparison of the critical dynamic buckling of clamped FGM shallow spherical shell ($P_{cr}\times10^5$), ($R/h = 1000$, $r_0/R = 0.2$, $n = 1$) with the change of index k.

k		0	1	5	10
$P_{cr}(FM)$	Present	3.0135	2.4289	2.1254	1.9263
	Bich [14]	2.9995	2.3796	1.9767	1.8826

Table 1b
Comparison of the critical dynamic buckling of clamped FGM shallow spherical shell ($P_{cr}\times10^5$), ($k = 1$, $r_0/R = 0.2$, $n = 1$) with the change of R/h.

R/h		1000	1200	1500	2000
$P_{cr}(FM)$	Present	2.3812	2.0039	1.5108	1.1327
	Bich [14]	2.3796	1.9565	1.4789	1.1093

5.2. The nonlinear dynamic and vibration of the S-FGM shallow spherical shells

This section presents the illustrative results for Al/Al_2O_3 S-FGM shallow spherical shell with the properties: $E_m = 70GPa$, $\rho_m= 2702kg/m^3$, $E_c = 380GPa$, $\rho_c = 3800kg/m^3$, $\nu = 0.3$.

In the special cases, the nonlinear dynamic responses of axisymmetric S-FGM spherical shells acted on by the harmonic uniformly external pressure load $q(t) = Qsin(\Omega t)$ are obtained by solving Eq. (27) combined with the initial conditions.

The obtained results in Table 2 show the fundamental frequencies of natural vibration. Obviously the natural fundamental frequencies of axisymmetric S-FGM spherical shells observed to be dependent on the constituent volume fractions k, they increases when increasing the power index k. This is completely reasonable due to the lower value of the elasticity modulus of the metal constituent in comparison with the ceramic. Especially, the fundamental frequency of immovable spherical shell is much greater than one of freely movable spherical shell.

Effect of volume-fraction index k and foundation on critical dynamic buckling of axisymmetric S-FGM spherical shells are showed in Table 3. Clearly, the critical dynamic buckling of shell decrease when the volume-fraction index increases. As can be also observed, the critical dynamic buckling load of shells with EF is greater than one of shells without EF. This was described in more detail as below in the Fig. 2.

Fig. 3 shows the effect of ratio of radius of curvature and base radius R/r_0 on critical dynamic buckling of axisymmetric S-FGM spherical shells in the case without EF. The obtained result shows that critical dynamic buckling of the shells is very sensitive with change of ratio R/r_0 characterizing the shallowness of the shell, the instability region is clearly recognized with small R/r_0 ratio but it is very difficult to define that when the R/r_0 ratio increases.

Effect of temperature on dynamic buckling of immovable axisymmetric S-FGM spherical shells is presented in Table 4. Easy to see that, when the temperature increases, the critical dynamic buckling of shell increases, however this influence is not too strong on dynamic buckling of immovable axisymmetric S-FGM spherical shells.

Table 2
Fundamental frequencies of axisymmetric S-FGM shallow spherical shells ($\times10^3$) ($R = 3$, $R/h = 100$, $k_1 = 5.10^7N/m^3$, $k_2 = 10^5N/m$) (with and without effect elastic foundation (EF)).

		Freely movable		Immovable	
		Without EF	With EF	Without EF	With EF
$R/r_0 = 3$	$k = 0$	0.9118	1.0845	6.8816	6.9066
	$k = 1$	1.7794	1.8854	13.0992	13.1140
	$k = 5$	2.2130	2.3068	16.5304	16.5432
	$k = \infty$	2.4119	2.5024	18.2033	18.2155
$R/r_0 = 5$	$k = 0$	0.7766	0.9000	5.3142	5.3337
	$k = 1$	1.6181	1.6886	10.1326	10.1440
	$k = 5$	1.9394	2.0039	12.7740	12.7840
	$k = \infty$	2.5042	2.1181	14.0572	14.0667

Table 3
Dynamic buckling of axisymmetric S-FGM shallow spherical shell. ($\times10^8N/m^2$) ($R = 3$, $R/h = 100$, $k_1 = 5.10^7N/m^3$, $k_2 = 10^6N/m$, $c = 10^9\ N/m\ s$).

		Freely movable		Immovable	
		Without EF	With EF	Without EF	With EF
$R/r_0 = 2$	$k = 0$	0.0870	0.1489	1.7306	1.7887
	$k = 1$	0.1725	0.2208	5.4976	5.5553
	$k = 5$	0.2254	0.2723	8.0062	8.0639
	$k = \infty$	0.2510	0.2972	9.2590	9.3167
$R/r_0 = 3$	$k = 0$	0.0630	0.0974	1.6611	1.6888
	$k = 1$	0.1201	0.1500	5.2984	5.3260
	$k = 5$	0.1481	0.1760	7.7141	7.7418
	$k = \infty$	0.1576	0.1846	8.9183	8.9458

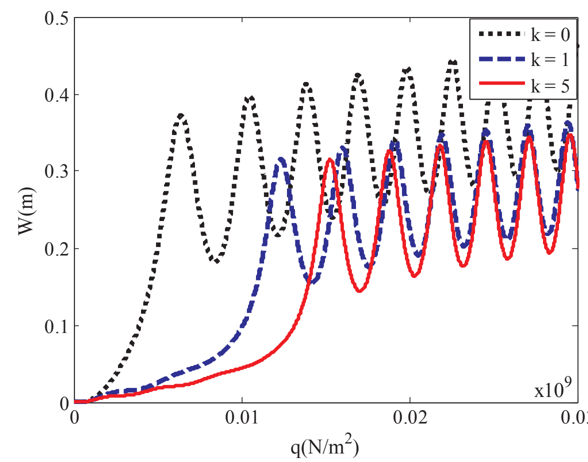

Fig. 2. Effect of index k on critical dynamic buckling of axisymmetric S-FGM spherical shells without EF.

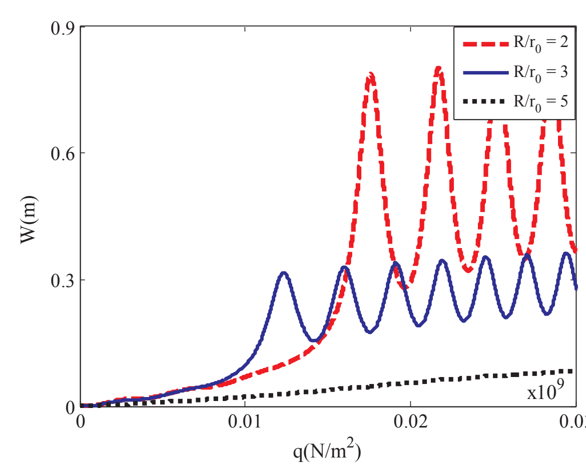

Fig. 3. Effect of ratio R/r_0 on critical dynamic buckling of axisymmetric S-FGM spherical shells without EF.

Table 4
Effect of temperature on dynamic buckling of immovable axisymmetric S-FGM shallow spherical shells ($\times10^8N/m^2$) $k = 1$,$R/h = 100$, $k_1 = 5.10^7N/m^3$, $k_2 = 10^6N/m$.

	$\Delta T = 0$	$\Delta T = 100$	$\Delta T = 300$	$\Delta T = 500$
$R/r_0 = 3$	5.3260	5.3702	5.4480	5.5258
$R/r_0 = 5$	4.9855	5.0259	5.1062	5.1821

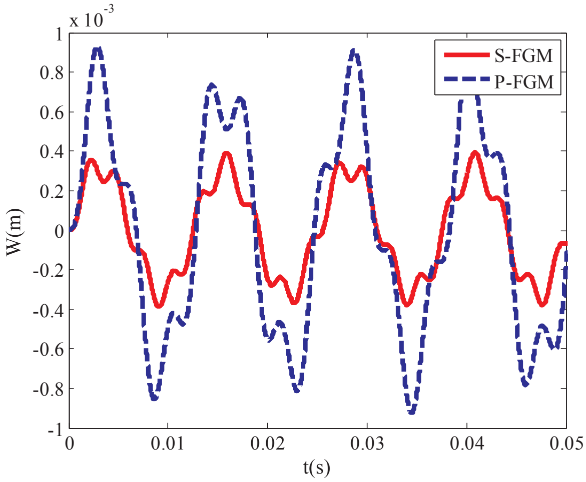

Fig. 4. Comparisons of the amplitude fluctuation for axisymmetric S-FGM with P-FGM shallow spherical shell (Freely movable) ($q = 10^5 \sin(500t)$).

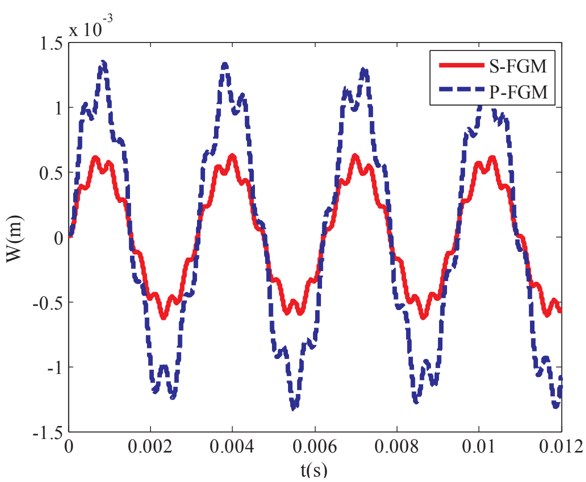

Fig. 5. Comparisons of the amplitude fluctuation for axisymmetric S-FGM with P-FGM shallow spherical shell (Immovable) ($q = 10^7 \sin(2000t)$).

To validate the proposed approach, comparison of the amplitude fluctuation for axisymmetric S-FGM shallow spherical shells (metal-ceramic-metal layers) with axisymmetric P-FGM shallow spherical shells (ceramic-metal layers) with the same geometrical dimensions ($k = 3$, $r_0 = 1\ (m)$, $R/r_0 = 3$, $R/h = 100$) is shown in Figs. 4 and 5. As can be seen, the amplitude fluctuation of the P-FGM shell is higher than the amplitude fluctuation of the S-FGM shell layers. Notice that, for the P-FGM studies, the nonlinear dynamic and vibration of this shell have been shown in [22] by Bich et al. and in this comparison, Figs. 4 and 5 show that the dynamic response of axisymmetric S-FGM shallow spherical shells is better than P- axisymmetric FGM shallow spherical shells.

In general cases (the non-axisymmetric spherical shells), the nonlinear dynamic responses of the S-FGM spherical shells acted on by the harmonic uniformly external pressure load $q(t) = Qsin(\Omega t)$ are obtained by solving Eq. (30) combined with the initial conditions and by use of the Runge–Kutta method. Fig. 6 shows nonlinear responses of the S-FGM spherical shells with various Sigmoid law indices subjected to the excited load of magnitude $Q = 2000\, N/m$ and frequency $\Omega = 300(s)$ different far from the natural frequencies of the S-FGM shells with $k = 0,\ 1,\ 5$. From the obtained results we can see that amplitudes of nonlinear vibration of the S-FGM shells increase when increasing the power law index k but frequencies decrease when increasing k. The nonlinear dynamic responses perform the phenomenon of periodic

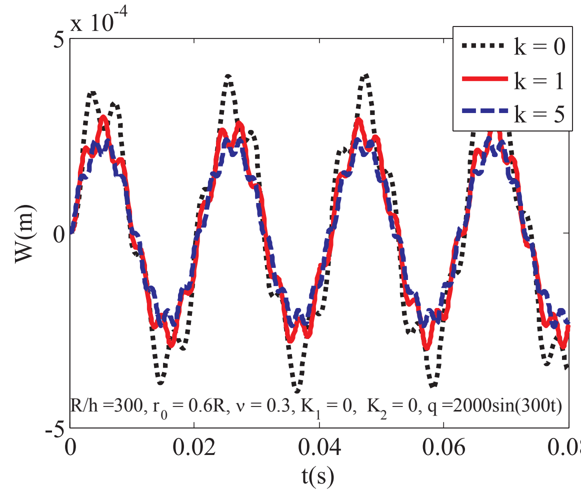

Fig. 6. Effect of power law index k on the nonlinear dynamic response of the S-FGM shallow spherical shell.

cycles.

Fig. 7 consider the effect of harmonic uniform load with amplitudes $Q = 1000(N/m^2)$, $Q = 2000\ (N/m^2)$ and $Q = 3000\ (N/m^2)$ on the nonlinear dynamic response of the S-FGM shallow spherical shell. From figure, it is seen that the nonlinear dynamic amplitude of the S-FGM shallow spherical shell is considerably increased when excitation force amplitude Q increases.

Fig. 8 shows the variation of nonlinear dynamic response amplitudes of the S-FGM shallow spherical shell with various values of the thickness h without EF ($K_1 = K_2 = 0$). Clearly, the thickness played a positive role on the dynamic response of the shell: the higher the h, the lower the amplitude of deflection.

The influence of radius R on the nonlinear dynamic response amplitudes of the S-FGM shallow spherical shell are presented in Fig. 9. As expected, the reduction of the radius R makes the amplitudes of nonlinear vibration of the S-FGM shell decrease.

Figs. 10 and 11 consider the effect of coefficients k_1, k_2 of the Winkler and Pasternak foundations, respectively, on the nonlinear dynamic response of the S-FGM shallow spherical shell. Obviously, the S-FGM shallow spherical shell amplitude fluctuation became considerably lower due to the support of the EF also. In addition, the shear layer stiffness k_2 of the Pasternak foundation model has a more pronounced influence than the modulus k_1 of the Winkler model on the nonlinear dynamic response of the S-FGM shallow spherical shell.

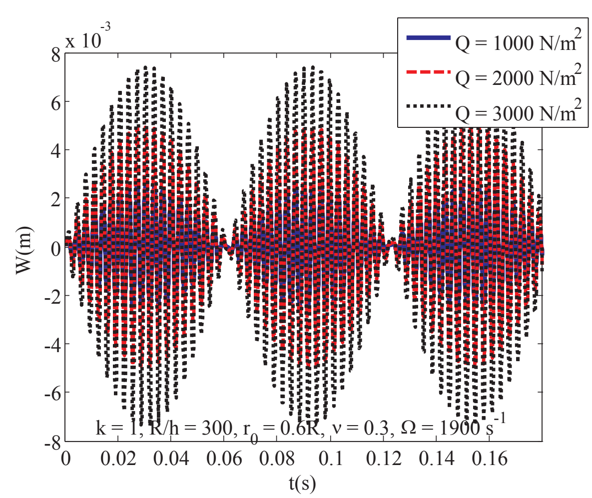

Fig. 7. Effect of the excitation force Q on the nonlinear dynamic response of the S-FGM shallow spherical shell.

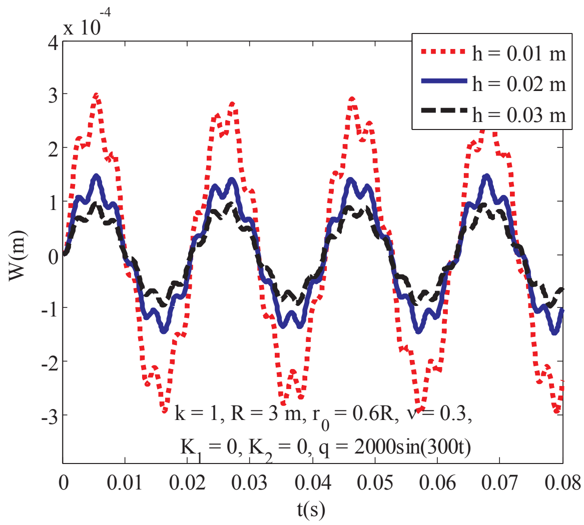

Fig. 8. Effect of the thickness on the nonlinear dynamic response of the S-FGM shallow spherical shell.

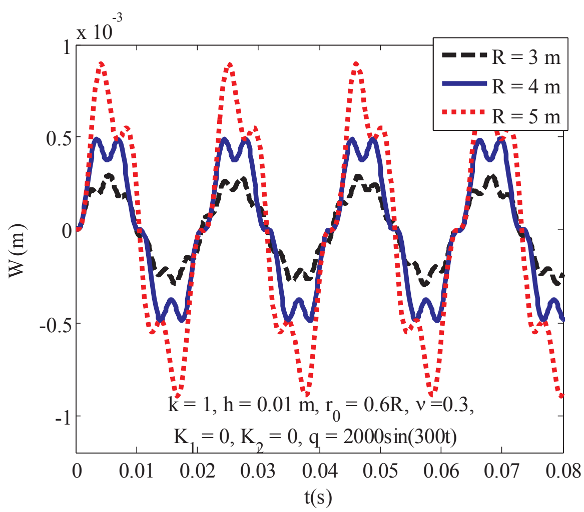

Fig. 9. Effect of the radius R on the nonlinear dynamic response of the S-FGM shallow spherical shell.

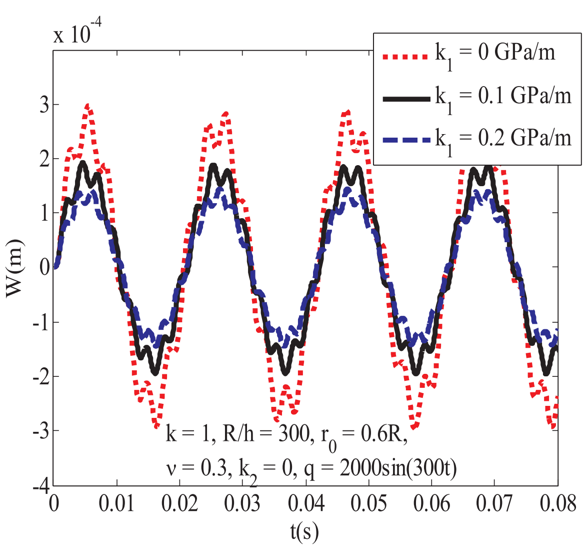

Fig. 10. Effect of the Winkler foundation on the nonlinear dynamic response of the S-FGM shallow spherical shell.

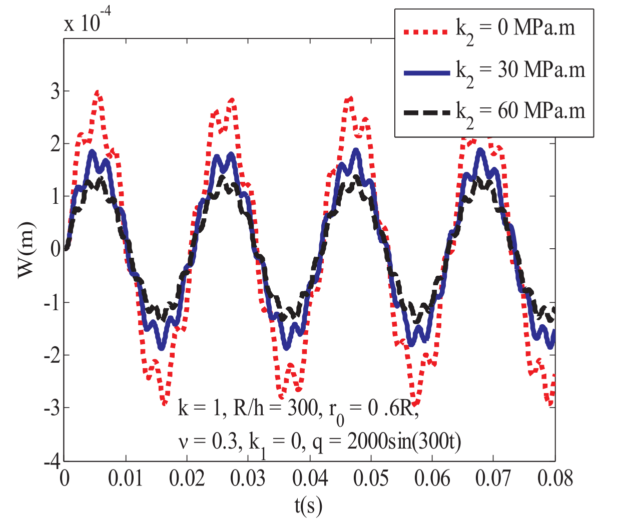

Fig. 11. Effect of the Pasternak foundation on the nonlinear dynamic response of the S-FGM shallow spherical shell.

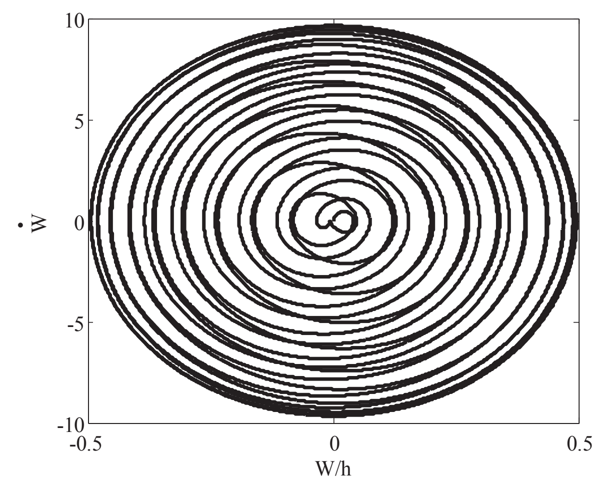

Fig. 12. Deflection–velocity relation of the S-FGM shallow spherical shell without EF ($R/a = 3$, $q = 2000\sin(1900t)$).

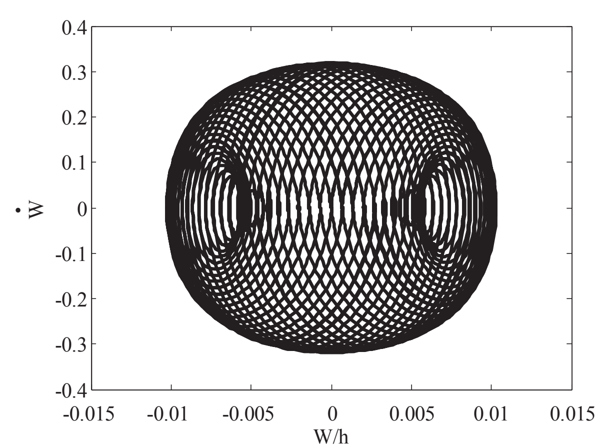

Fig. 13. Deflection–velocity relation of the S-FGM shallow spherical shell without EF ($R/a = 3$, $q = 2000\sin(6960t)$).

The deflection–velocity relation has the closed curve form as in Fig. 12 (when exited frequency is near natural frequency) and in Fig. 13 (when exited frequency is from to natural frequency).

Fig. 14 shows the effect of external force Q on the frequency-amplitude relations of nonlinear vibration of the S-FGM shallow

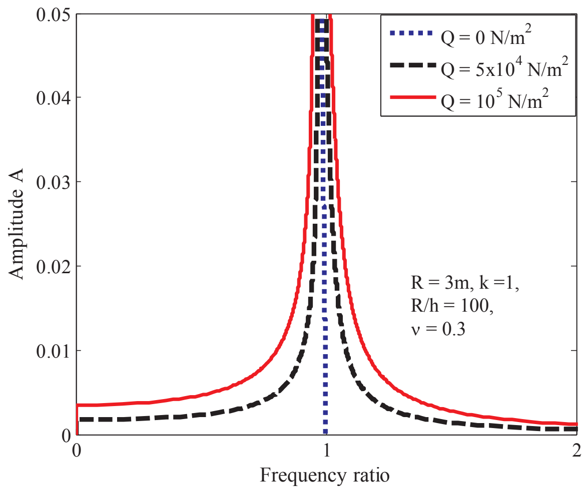

Fig. 14. Effect of external force Q on frequency-amplitude curves of the S-FGM shallow spherical shell without EF.

spherical shell. As can be observed, as the amplitude of external force increases, the amplitude-frequency curves of forced vibration asymptote that of the free vibration.

6. Conclusions

This paper investigated the nonlinear dynamic analysis and vibration for S-FGM shallow spherical shells (in two cases: the non-axisymmetric and the axisymmetric shells) subjected to mechanical loads including the effects of temperature and resting on an elastic foundations with different types of boundary conditions. The volume fractions of the metal and ceramic are applied by Sigmoid-law distribution with ceramic-metal-ceramic layers. Formulation for deformed spherical shells is based on the classical shell theory with geometrical nonlinearity is incorporated. The approximate solutions are assumed to satisfy the different types of boundary conditions. By using Galerkin method and four order Runge-Kutta method, the nonlinear dynamic response of the shells is analyzed and the results are illustrated in table form and graphic form. The critical dynamic buckling of shells is determined by according to the Budiansky–Roth criterion. The influence of inhomogeneous parameters, dimensional parameters, EF, types of boundary conditions, mechanical loads and temperature on the nonlinear dynamic analysis and vibration of the S-FGM shallow spherical shells are examined in detail.

Some conclusions can be obtained from the present analysis:

– The dynamic response of axisymmetric S-FGM is better than axisymmetric P- FGM shallow spherical shells.
– The natural fundamental frequencies of axisymmetric S-FGM spherical shells and the amplitudes of nonlinear vibration of the S-FGM shells increase when increasing the volume fractions k, but the critical dynamic buckling of axisymmetric S-FGM spherical shells and the frequencies of nonlinear vibration of the S-FGM shells decrease when increasing k.
– The elastic foundation, inhomogeneous parameters, dimensional parameters are positive influence on the nonlinear dynamic and vibration of the S-FGM shallow spherical shells (axisymmetric and non-axisymmetric).
– The nonlinear dynamic amplitude of the S-FGM shallow spherical shell is considerably increased when excitation force amplitude Q increases.
– The influence of temperature is not too strong on dynamic buckling of immovable axisymmetric S-FGM spherical shells.

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