

# Performance Analysis of Frequency Reuse for PPP Networks in Composite Rayleigh–Lognormal Fading Channel

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**Abstract** Fractional frequency reuse (FFR) is an effective intercell interference coordination technique that mitigates intercell interference and improves users' performance, especially who suffer low signal-to-interference–noise ratios. To examine the performance of FFR algorithms, a Poisson point process network model in which base stations and users are distributed as a Poisson process has been recently developed to replace the traditional grid model. The average coverage probability of a typical user that connects to the closest base station, however, is only considered in Raleigh fading and then its closed-form is only found in case of high transmission signal-to-noise (SNR) or neglecting noise. This paper focuses on deriving a closed-form expression for the average network coverage probability in composite Rayleigh–Lognormal for both low and high SNR.

**Keywords** Coverage probability · Ergodic capacity · Random cellular network · Rayleigh–Lognormal · Gauss–Hermite quadrature · Gauss–Legendre quadrature · Strict frequency reuse · Soft frequency reuse

## 1 Introduction

In orthogonal frequency division multiple (OFDM) multi-cell networks, the main factor that directly impacts on the system performance is intercell interference that causes by the use of the same frequency band in adjacent cells. Intercell interference coordination (ICIC) [1] is promised as a technique that can significantly mitigate the intercell interference then improve the network performance, especially for users suffering low signal-to-interference–noise ratio (SINR). Fractional frequency reuse (FFR) which basically divides the

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available resource blocks in several parts has been proposed as an effective ICIC technique. Under FFR, the base stations use some pre-defined frequency reuse algorithms to allocate resource blocks to users in an effort to improve spectrum efficiency and reduce intercell interference. There are two well-known FFR deployment algorithms, called strict frequency reuse (Strict FR) and soft frequency reuse (Soft FR).

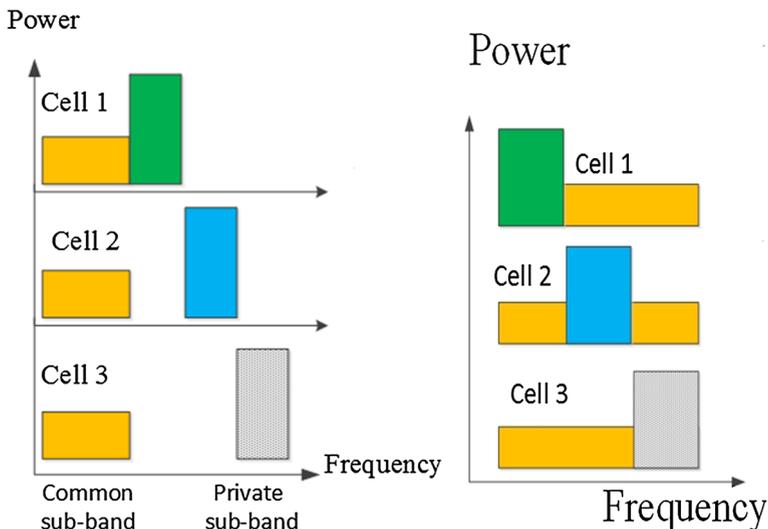
*Strict FR*: in strict FR with reuse factor  $\Delta = 3$  as in Fig. 1, the whole bandwidth is divided into one common sub-band group and  $\Delta$  private sub-band groups. The common sub-band group is allocated to cell-center users with a low power in every cell while each private sub-band group is allocated to the cell-edge users with a high power. In this scheme, the cell-center users, additionally, do not share their own resource blocks to the cell-edge users, then if there are  $N$  interfering cells that effect on interior users, there are only  $\frac{N}{\Delta}$  for the case of cell-center users. The interference on both cell-edge and cell-center users are, hence, minimized.

*Soft FR*: soft FR is a modification of the Strict FR as in Fig. 1 in which the cell-center users are allowed to share the allocated sub-band with cell-edge users in adjacent cells. Because of sharing the resource, Soft FR outperforms Strict FR in terms of system performance and spectrum efficiency. However, Soft FR creates more intercell interference for both cell-center and cell-edge users [2].

## 1.1 Related Work and Contributions

In realistic mobile radio scenarios in urban areas, the multipath effect at the mobile receiver due to scattering from local scatters such as buildings in the neighborhood of the receiver causes a fast fading while the variation in the terrain configuration between the base-station and the mobile receiver causes a slow shadowing.

Therefore, the mobile radio signal envelope is usually composed of a *small scale multipath fading* component superimposed on a *larger scale* or *slower shadowing* component. It is well known that the signal envelope of the multipath component can be modeled as a Rayleigh distributed random variable (RV), and its power can be modeled as



**Fig. 1** Resource allocation in Strict FR (left) and Soft FR (right)

an exponential RV. Thus the path power gain has a mixed Rayleigh–Lognormal distribution which is also known as the Suzuki fading distribution model [3].

The traditional grid network model with deterministic BS locations is no longer accurate to evaluate the performance of multi-cell wireless networks [4]. Poisson point process (PPP) network model [4] in which the BS locations are modeled as a homogeneous spatial Poisson point process has been developed as a new accurate and flexible tractable model for cellular networks. As in well-known network model, a typical user is allowed to associate with the closest or the strongest BS. In strongest model, a user measures signal-to-interference–noise ratio (SINR) from several BSs and selects the BS that provides the highest SINR while in closest model, the distance between a user and BSs is estimated, and the BS which is nearest to the user is selected.

There are two common methods to evaluate the performance of PPP network, called coverage probability approach [4] and moment generating function (MGF) approach [5].

Coverage probability approach was proposed to calculate the coverage probability and capacity of a typical user that associates with its nearest base station [4, 6, 7], then developed for PPP network enabling frequency reuse [8]. Due to the complexity of mathematical manipulations, the mathematical expression stopped at two layer integrals [8]. Author in [9] derived a technique that can reduce the complexity of this approach but there was confusion between the closet and the strongest PPP model. In these frameworks, the closed-form expressions were only found in a case of ignoring Gaussian noise and only in Rayleigh fading. Then, closed-form expression for coverage probability still needs to be investigated and developed for composite Rayleigh–Lognormal fading channel.

Moment generating function (MGF) approach was proposed in [5] to avoid the complexity of coverage probability approach. By using this approach, the authors derived the average capacity of a user in simple PPP network in generalized fading channels. The final equations, however, were not exactly simple because they contained the Gauss hypergeometric function [10] which is expressed as an integral. This approach, additionally, is difficult to develop for a network using frequency reuse which is considerably complex than a simple PPP network. Hence, coverage probability approach is still considered as an appropriate method to examine the performance of PPP network.

Some works that evaluated the effect of Rayleigh and shadowing was considered in [11, 12]. However, in [11], shadowing was not incorporated in channel gain and can be constant when the origin PPP model is rescaled. Instead of rescaling network model, authors in [12] introduced a new approach to derive the coverage probability mathematical expression for zero noise PPP network (neglecting noise). However, these approaches did not consider frequency reuse and the given formulas still stopped at the two layer integrals.

This paper bases on the coverage probability approach to derive a closed-form expression for coverage probabilities of a typical user in the closest PPP network model enabling Strict and Soft FR. These coverage probabilities were produced in [8], but due to some reasons, they significantly differ from the results presented in this paper. Firstly, instead of dealing with only Rayleigh fading channel, the paper deals with composite Rayleigh–Lognormal fading channels. The second reason relates to the modeling of Soft FR algorithms, the cell-edge sub-band in [8] is randomly selected from the whole bandwidth, while in this paper, the cell-edge sub-band is only chosen from the determined cell-edge sub-band group [1].

## 2 System Model

### 2.1 Network Topology

In this paper, a PPP cellular network (as shown in Fig. 2) in which the locations of BSs are distributed as a homogeneous spatial Poisson point process (PPP) with density  $\lambda$  is considered. Users are randomly located according to an independent stationary point process in a Voronoi cell and has connection with the closest BS [4].

With loss of generality, a typical user is assumed to be located at the origin and served by BS at distance  $r$ . The probability density function (PDF) of  $r$  is given by:

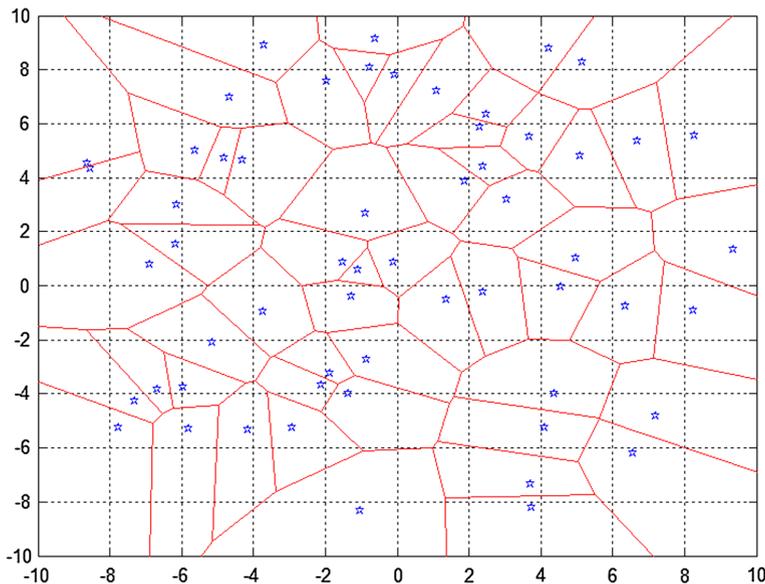
$$f_r(r) = 2\pi\lambda e^{-\pi\lambda r^2} \quad (1)$$

In this analysis, all BSs is assumed transmit continuously in all time slots and at constant power, although if their operation is interrupted with coefficient  $\xi$ , then the density of BSs should be changed from  $\lambda$  to  $(1 - \xi)\lambda$  [7].

### 2.2 Channel Model

In downlink cellular network, the signal transmitted over distance  $r$  suffers free-space path loss, fast fading as well as slow fading.

The free-space path loss (FSPL) can be defined as the loss of signal strength of an electromagnetic signal originated from the line of sight path through free space without any obstacles that cause any reflection or diffraction. For a receiver at distance  $r$  from the transmitter, the received power is  $P_r = \zeta P r^{-\alpha}$  in which  $\alpha$  is path-loss coefficient;  $P$  and  $\zeta$  are standard transmission power of a base station and power adjustment coefficient, respectively,  $\zeta > 0$ .



**Fig. 2** An example of PPP network model with  $\lambda = 0.2$

The probability density function (PDF) of power gain  $g$  of signal experiencing Rayleigh and Lognormal fading is found from the probability density function (PDF) of the product two cascade channels [3].

$$f_{R-Ln}(g) = \int_0^\infty \frac{1}{x} \exp\left(-\frac{g}{x}\right) \frac{1}{x\sigma_z\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu_z)^2}{2\sigma_z^2}\right) dx \tag{2}$$

Let

$$t = \frac{10 \log_{10} g - \mu_z}{\sqrt{2}\sigma_z} \Rightarrow g = 10^{(\sqrt{2}\sigma_z t + \mu_z)/10} = \gamma(t)$$

then

$$f_{R-Ln}(g) = \int_0^\infty \frac{1}{\sqrt{\pi}} \frac{1}{\gamma(t)} \exp\left(-\frac{g}{\gamma(t)}\right) \exp(-t^2) dx$$

By using Gauss–Hermite expansion [10], the PDF equals

$$f_{R-Ln}(g) = \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \frac{1}{\gamma(a_n)} \exp\left(-\frac{g}{\gamma(a_n)}\right)$$

in which

$w_n$  and  $a_n$  are, respectively, the weights and the abscissas of the Gauss–Hermite polynomial. The approximation becomes more accurate with increasing approximation order  $N_p$ . For sufficient approximation,  $N_p = 12$  is used.

$\gamma(a_n) = 10^{(\sqrt{2}\sigma_z a_n + \mu_z)/10}$ ;  $\mu_z$  and  $\sigma_z$  are mean and variance of Rayleigh–Lognormal random variable.

Then, its cumulative density function is given by

$$F_{R-Ln}(g) = \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \left[ 1 - \exp\left(-\frac{g}{\gamma(a_n)}\right) \right] \tag{3}$$

The associated signal power which a normal user receives from its serving BS at distance  $r$  in composite Rayleigh–Lognormal fading is:

$$S = \zeta P g r^{-\alpha} \tag{4}$$

### 2.3 Signal-to-Interference–Noise Ratio (SINR)

The set of interfering BS is denoted as  $\theta$ ;  $r_u$  and  $g_u$  are the distance and channel gain from a user to an interfering BS respectively. The interfering base station transmits at power  $P_u = \rho P$  ( $\rho > 0$ ). Since a user connect to the closest BS,  $r_u > r$ . The intercell interference that causes a user is obtained by

$$I_\theta = \sum_{\theta} \rho P g_u r_u^{-\alpha} \tag{5}$$

Combining (4) and (5), the received instantaneous SINR at a normal user is found from Eq. (6)

$$SINR = \frac{Pg r^{-\alpha}}{\sigma^2 + I_\theta} \quad (6)$$

## 2.4 Fractional Frequency Reuse

In the nearest model of random cellular network enabling fractional frequency reuse, a mobile user computes SINR from its serving base station and then compares with the frequency reuse threshold  $T$ . If the measured SINR is less than the threshold, then the base station randomly selects an available sub-band which is the private sub-band within  $\Delta$  cells to serve this user. The user, hence, experiences new power gain  $g'$  and new interference  $I'_\theta$ . However, the user still connects to the nearest base station and experiences the same propagation path loss  $r^{-\alpha}$ .

In *Strict FR*, the cell-center users do not share spectrum with the cell-edge users, then the interference on the cell-edge user in this case is defined as

$$I_\theta = \sum_{u \in \theta_{edge}} P_E g_u r_u^{-\alpha}$$

In *Soft FR*, the base station can reuse any sub-band then the intercell interference at a typical user is caused by the base stations in two groups  $\theta_{center}$  and  $\theta_{edge}$  which consist all base stations serving cell-edge and cell-center users on the same sub-band with user  $u$ . To classify two groups,  $\beta$  is defined as the ratio between the transmission power on the cell-center sub-band  $P$  and on the cell-edge sub-band  $P_E$  [8]. Then the total interference equals

$$I_\theta = \sum_{z \in \theta_{center}} P g_{zu} r_{zu}^{-\alpha} + \sum_{v \in \theta_{edge}} \beta P g_{vu} r_{vu}^{-\alpha} \quad (7)$$

We assume that the available sub-bands, given by  $N$ , are divided into two non-overlapping sub-band groups denoted by  $N_c$  and  $N_E$  which are located to cell-edge and cell-center users, respectively.

## 3 Coverage Probability

The coverage probability  $P_c$  of a typical user for a given threshold  $T_c$  is defined as the probability of event in which the SINR is larger than the threshold. [4] proved that  $P_c$  is a function of SINR threshold  $T_c$ , BS density  $\lambda$ , attenuation coefficient  $\alpha$  and number of the allocated sub-bands  $N$ . For simple expression,  $P_c$  can be written as a function of threshold  $T$ :

$$P_c(T_c, N, \zeta, \rho, r) = \mathbb{P}(SINR > T_c) \quad (8)$$

in which  $r$  is the distance from the user to its serving base station whose transmit power is  $\zeta P$ ,  $\rho P$  is the transmit power of the interfering base station. This coverage probability is exactly the complementary part of the CDF of SINR, since the CDF is defined as  $F_c = \mathbb{P}(SINR < T_c)$ .

**Theorem 1** *The coverage probability of a typical user on a particular sub-band with transmit power  $\zeta P$  in Rayleigh–Lognormal fading when BSs are distributed as PPP with density  $\lambda$  and are allocated randomly  $N$  sub-bands is given by*

$$P_c(T_c, N, \zeta, \rho, r) = \sum_{n=1}^{N_p} \frac{W_n}{\sqrt{\pi}} e^{-\frac{T_c \zeta}{\gamma(a_n) SNR} r^\alpha} e^{-\frac{\pi \lambda^2}{N} f_{SFR}(T_c, \zeta, \rho, n)} \tag{9}$$

where  $SNR = \frac{P}{\sigma^2}$  is the signal-to-noise ratio at the transmitter,  $C = T_c \frac{\rho}{\zeta} \frac{\gamma(a_{n1})}{\gamma(a_n)}$ ,

$$f_{SFR}(T_c, \zeta, \rho, n) = \sum_{n_1=1}^{N_p} \frac{W_{n_1}}{\sqrt{\pi}} \left( \frac{1}{\alpha} C^{\frac{1}{\alpha}} \frac{\pi}{\sin\left(\frac{\pi(\alpha-1)}{\alpha}\right)} - \sum_{n_{GL}=1}^{N_{GL}} c_i \frac{C}{2 C + \left(\frac{x_i+1}{2}\right)^\alpha} \right) \tag{10}$$

where  $c_i$  and  $x_i$  are weights and nodes of Gauss–Legendre rule respectively with order  $N_{GL}$ . In this paper,  $N_{GL} = 10$  is sufficient for accurate computation.

*Proof* See Appendix 1.

In a case of both serving and interfering base stations use the same power level to transmit,  $\zeta = \rho$ , then the coverage probability can be written in simple form  $P_c(T_c, N, r)$  and then  $C = T_c \frac{\gamma(a_{n1})}{\gamma(a_n)}$ . This result is consistent with the result in [13].

It is observed from Theorem 1 that the coverage probability is inversely proportional to exponential function of  $\frac{1}{SNR}$  and  $\frac{1}{N}$ . It is reminded that the probability in which a base station causes interference on a typical user is inversely proportional with the number of available sub-bands and then it reaches 0 when  $N \rightarrow \infty$ . Subsequently, the total interference  $I_\theta = 0$  when  $N \rightarrow \infty$ . In this case, the coverage probability is re-defined as the probability in which the desired signal power is larger than the threshold and is found from:

$$P_c(T_c, N, r) = \sum_{n=1}^{N_p} \frac{W_n}{\sqrt{\pi}} e^{-\frac{T_c \zeta}{\gamma(a_n) SNR} r^\alpha}$$

The average coverage probability is obtained by integrating coverage probability  $P_c(T, N, \zeta, \rho, r)$  over entire PPP network with variable  $r > 0$ .

$$\bar{P}_c(T_c, N, \zeta, \rho) = \int_0^\infty 2\pi\lambda e^{-\pi\lambda r^2} P_c(T_c, N, \zeta, \rho, r) d$$

Letting  $r = \frac{t}{1-x}$ , and using Gauss–Legendre, the average coverage probability approximately equals

$$\bar{P}_c(T_c, N, \zeta, \rho) = 4\pi\lambda \sum_{i=1}^{N_{GL}} \frac{c_i (x_i + 1)}{(1 - x_i)^3} e^{-\pi\lambda \left(\frac{x_i+1}{1-x_i}\right)^2} P_c\left(T_c \middle| r = \frac{x_i + 1}{1 - x_i}\right) \tag{11}$$

Hence, the outage probability in which the associated SINR is less than the threshold is given by:

$$P_0(T_c, N, \zeta, \rho, r) = 1 - \sum_{n=1}^{N_p} \frac{W_n}{\sqrt{\pi}} e^{-\frac{T_c \zeta}{\gamma(a_n) SNR} r^\alpha} e^{-\frac{\pi \lambda^2}{N} f_{SFR}(T, n)} \tag{12}$$

and its average is obtained by

$$\bar{P}_o(T_c, N, \zeta, \rho) = 4\pi\lambda \sum_{i=1}^{N_{GL}} \frac{c_i(x_i + 1)}{(1 - x_i)^3} e^{-\pi\lambda \left(\frac{x_i+1}{1-x_i}\right)^2} \left[ 1 - P_c\left(T_c | r = \frac{x_i + 1}{1 - x_i}\right) \right] \quad (13)$$

When  $\zeta = \rho$ , these probabilities can be written in simple forms such as  $P_o(T_c, N, r)$  and  $\bar{P}_o(T_c, N)$  respectively.

### 3.1 Strict Frequency Reuse

In strict frequency reuse, the user whose SINR on the common sub-band is smaller than a threshold  $T_{FR}$  is served on the cell-edge sub-band which is the private sub-band within a group of  $\Delta$  cells. Hence, the cell-center (cell-edge) user  $u$  is only affected by interference from base stations serving other cell-center (cell-edge) users on the same sub-band with user  $u$ . The densities of interfering base stations in this case are  $\lambda$  and  $\frac{\lambda}{\Delta}$ , respectively.

**Lemma 1** *The probability in which a particular base station causes interference on a typical user does not depend on frequency reuse factor.*

*Proof* Since the cell-edge sub-band is partitioned into  $\Delta$  equal parts, the cell-edge users in each cell are only allowed to use  $\frac{N_c}{\Delta}$  sub-bands. It is noticed that when BSs have PPP distribution with density  $\lambda$ , the density of BSs which causes interference on cell-edge users in this case is only  $\lambda' = \frac{\lambda}{\Delta}$ . Then, it is seen that the probability in which two base stations transmit on the same cell-edge sub-band at the same time is increased by  $\Delta$  times after partitioning the cell-edge sub-bands and also decreased by  $\Delta$  times due to the decrease in the density of interfering base stations. Subsequently, this probability is consistent with the changes of frequency reuse factor  $\Delta$ . This *Lemma* is also given by mathematical manipulations in *Theorem 1*. Then effective intercell interference on the cell-edge user is given by

$$I_\theta = \sum_{u \in \theta} \frac{\beta}{N_E} P_c g_u r_u^{-\alpha}$$

**Theorem 2** (Strict FR) *The average coverage probability of the cell-edge user is given by:*

$$P_{StrE}(T, T_c) = \frac{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_c\left(T_c, N_E, \beta, \frac{\beta}{N_E}, x_i\right) P_o(T, N_C, x_i)}{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T, N_C, x_i)} \quad (14)$$

In which  $P_c\left(T_c, N_E, \beta, \frac{\beta}{N_E}, x_i\right)$  and  $P_o(T, N_C, x_i)$  are defined in (8) and (12), and

$$f(c_i, x_i) = \frac{c_i(x_i+1)}{(1-x_i)^3} e^{-\pi\lambda \left(\frac{x_i+1}{1-x_i}\right)^2}$$

*Proof* This Theorem can be proved by using the results of Appendices 1, 2 with  $\eta = \frac{\beta}{N_E}$  for the numerator and  $\eta = \beta = 1$  for the denominator.

Due to the fact that the intercell interference does not depend on  $\Delta$  (Lemma 1), the coverage probability in Eq. (14) does not depend on the frequency reuse factor  $\Delta$ . This result differs from the result in [8] because authors in [8] assumed that in Strict FR, all base stations can produce interference on the cell-edge user which is only valid in the case of  $\Delta = 1$ .

**Theorem 3** (Strict FR) *The average coverage probability of the cell-center user is given by equation:*

$$P_{StrE}(T, T_c) = \frac{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_c(\max(T, T_c), N_c, x_i)}{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_c(T, N_c, x_i)} \tag{15}$$

*Proof* The average coverage probability of the cell-center user is defined as below

$$\begin{aligned} P_{StrE}(T, T_c) &= \mathbb{P}\left(\frac{Pgr^{-\alpha}}{\sigma^2 + I_\theta} > T_c \mid \frac{Pgr^{-\alpha}}{\sigma^2 + I_\theta} > T\right) \\ &= \frac{\mathbb{P}\left(\frac{Pgr^{-\alpha}}{\sigma^2 + I_\theta} > \max(T_c, T)\right)}{\mathbb{P}\left(\frac{Pgr^{-\alpha}}{\sigma^2 + I_\theta} > T\right)} \end{aligned} \tag{16}$$

Combining (11) and (13), the Theorem 3 is proved.

### 3.2 Soft Frequency Reuse

Since, in Soft FR, the base stations are allowed to reuse the whole spectrum band, then a typical user may suffer interference from any neighboring base stations. The effective interference in this case is discussed in Lemma 2.

**Lemma 2** *For Soft FR, the effective intercell interference in Eq. (7) is*

$$I_\theta = \sum_{u \in \theta} \left( \frac{(\Delta - 1)}{\Delta} \frac{1}{N_E} + \frac{\beta}{\Delta N_C} \right) P_c g_u r_u^{-\alpha} \tag{17}$$

*Proof* The probabilities in which a typical user is served on the cell-edge and cell-center sub-bands are  $\frac{1}{N_E}$  and  $\frac{1}{N_C}$ , respectively.

When reuse factor  $\Delta$  is used, the densities of base stations in two groups  $\theta_{center}$  and  $\theta_{edge}$  that cause interference on the cell-edge user are  $\lambda_c = \frac{(\Delta-1)\lambda}{\Delta}$  and  $\lambda_E = \frac{\lambda}{\Delta}$ .

Then if it is assumed that the densities of base stations in two interfering groups are the same and equal  $\lambda$ , then the probabilities of the event in which a base station transmitting to cell-edge and cell-center users on the same sub-band as user  $u$  are  $\frac{(\Delta-1)}{\Delta} \frac{1}{N_C}$  and  $\frac{1}{\Delta N_E}$ . The total interference in Eq. (7) can be archived as below:

$$I_\theta = \sum_{u \in \theta} \left( \frac{(\Delta - 1)}{\Delta} \frac{1}{N_C} + \frac{\beta}{\Delta N_E} \right) P_c g_u r_u^{-\alpha}$$

Denote  $\eta = \frac{(\Delta-1)}{\Delta} \frac{1}{N_E} + \frac{\beta}{\Delta N_C}$ . Then  $\eta$  is called the effective interference power factor of the cell-edge user. The concept of this parameter has been already defined in [8], without considering frequency reuse factor and assuming that cell-center and cell-edge users can use the whole  $N$  sub-bands.

For  $\Delta = 1$  or  $\beta \rightarrow \infty$ , the effective intercell interference approximately equals

$$I_{\theta} = \sum_{u \in \theta} \frac{\beta}{\Delta N_E} P_c g_u r_u^{-\alpha}$$

This is exactly the intercell interference that impacts on the cell-edge user in Strict FR. Hence, it is said that Strict FR is a special case of Soft FR.

**Theorem 4** (Soft FR) *The average probability of the cell-edge user obtained by:*

$$P_{SoftE}(T, T_c, \beta, \eta) = \frac{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_c(T_C, N_E, \beta, \eta, x_i) P_o(T, N_C, 1, \eta, x_i)}{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T_c, N_C, 1, \eta, x_i)} \tag{18}$$

where  $P_c(T_C, N_E, \beta, \eta, x_i)$  and  $P_o(T, N_E, 1, \eta, x_i)$  are defined in Eqs. (8) and (12).

*Proof* See Appendix 2.

**Theorem 5** (Soft FR) *The average coverage probability of the cell-center user obtained by:*

$$P_{SoftC}(T, T_c, \beta, \eta) = \frac{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_c(\max(T, T_c), N_C, 1, \eta, x_i)}{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T_c, N_C, 1, \eta, x_i)} \tag{19}$$

*Proof* The average coverage probability expression is given in a same way in Theorem 3.

The closed-form expressions for average coverage probability of cell-edge and cell-center users in Strict and Soft FR are presented in Eqs. (14–19). These results are expressed in form of finite sums that are significantly simple than the integrals in previous results [8]. Specially, the closed-forms are found in composite Rayleigh–Lognormal fading channels. These are the main contributions of this paper.

**Lemma 3** (Soft FR) *For  $\sigma^2 = 0$  or high SNR, then the average coverage probability becomes*

$$P_{SoftE}(T, T_c, \beta, \eta) = \frac{\sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \frac{1}{1 + \frac{1}{N_E} f_{SFR}(T_c, n, \beta, \eta)}}{1 - \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \frac{1}{1 + N_C f_{SFR}(T, N_C, n, 1, \eta)}} - \frac{\sum_{n=1}^{N_p} \sum_{m=1}^{N_p} \frac{w_n w_m}{\pi} \frac{1}{\frac{1}{N_E} f_{SFR}(T_c, \beta, \eta, n) + \frac{1}{N_C} f_{SFR}(T, 1, \eta, m)}}{1 - \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \frac{1}{1 + \frac{1}{N_C} f_{SFR}(T, N_C, n, 1, \eta)}} \tag{19a}$$

*Proof* See Appendix 3.

It is interesting to note that the average coverage probability in this case does not depend on the density of base station. This is due to the fact that an increase in number of base stations also increases the intercell interference. The similar conclusion for Rayleigh fading has been given in [4].

In the case of neglecting noise and in Rayleigh fading channels and due to  $\sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} = 1, P_{SofE}(T, T_c, \beta, \eta)$  formula is found by removing the finite sums in (19a).

### 4 Ergodic Capacity

The ergodic capacity is defined by Shannon formula

$$C(\lambda, \alpha) = \mathbb{E}[\ln(1 + SINR)]$$

in which the average is evaluated over entire PPP network and fading distribution. Then the CDF function of average rate can be written as below [6]

$$C(\lambda, \alpha) = \int_0^\infty \mathbb{P}[\ln(1 + SINR) > t] dt$$

Using Theorem 1, and Gauss–Legendre, the average rate of a typical user on a particular sub-band is approximated by

$$C(\lambda, \alpha) = \sum_{i=1}^{N_{GL}} \frac{2c_{1i}}{(1 - x_{1i})^2} P_c(z(x_{1i}), N, \zeta, \rho) \tag{20}$$

where  $c_{1i}$  and  $x_{1i}$  are weights and nodes of Gauss–Legendre rule with order  $N_{GL}$ ;  $z(x_{1i}) = \exp\left(\frac{x_{1i}+1}{1-x_{1i}}\right) - 1$  and  $P_c$  as defined in Eq. (10).

Due to the same approach deployed to calculate average capacity for a typical user in Soft FR and Strict FR, this session only presents the results for Soft FR. For cell-edge user, the average capacity is given by

$$C_{SE}(T, \lambda, \alpha) = \frac{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T, N_c, 1, \eta, x_i) \int_0^\infty P_c(t, N_E, \beta, \eta, x_i) dt}{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T, N_c, 1, \eta, x_i)} \tag{21}$$

Processing the same way in Theorem 1, the closed-form of average capacity of the cell-edge user is obtained by

$$C_{SE}(T, \lambda, \alpha) = \frac{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T, N_c, 1, \eta, x_i) \sum_{l=1}^{N_{GL}} P_c(z(x_{1i}), N_E, \beta, \eta, x_i)}{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T, N_c, 1, \eta, x_i)} \tag{22}$$

The average capacity of the cell-center user is given by

$$C_{SC}(T) = \frac{\sum_{i=1}^{N_{GL}} f(c_i, x_i) \sum_{l=1}^{N_{GL}} P_c(\max(z(x_{1i}), T_c), N_c, 1, \eta, x_i)}{\sum_{i=1}^{N_{GL}} f(c_i, x_i) P_o(T_c, N_c, 1, \eta, x_i)} \tag{23}$$

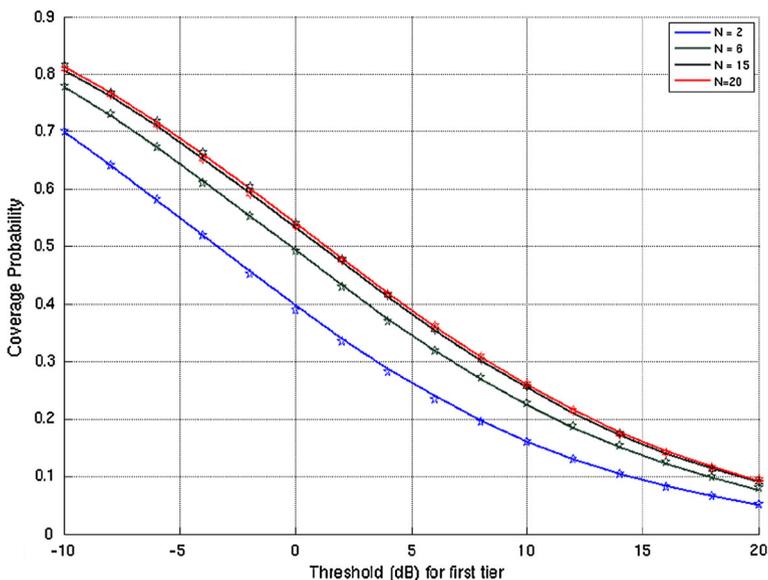
## 5 Simulation and Discussion

In this section, we use numerical method and Monte Carlo simulations to validate the theoretical analysis and to visualize the relationship between the average coverage probability and related parameters. In Figs. 3, 4, 5, the matching between the solid lines that present the theoretical analysis and the dotted lines that present the simulation results confirms the accuracy of the theoretical results.

It is assumed that base stations are allowed to share available 15 sub-bands which are classified into two groups. The first group containing 9 sub-bands is allocated to cell-center area, the second one containing 6 remaining sub-bands is allocated to cell-edge area. The base stations in both Strict and Soft FR are assumed to transmit at the same power level on cell-edge sub-bands which is  $\beta$  times larger than transmission power on cell-center sub-bands.

It is observed that the average coverage probability is proportional with number of sub-band  $N$  and reaches upper bound in high  $N$ . For  $SNR = 0$  dB, the average coverage probability with different values of  $N$  is given in Table 1.

For  $\Delta > 1$ , since the densities of interfering base station in both Soft and Strict FR are normalized (in Lemmas 1, 2) and equals  $\lambda$ , the worst user is affected by all neighboring base stations. However, in Soft FR, only some of them transmit at the cell-edge power level while in Strict FR, all interfering base stations transmit at the cell-center power level. Then, the cell-edge user in Strict FR experiences a higher interference power level than in Soft FR. This phenomenon is also shown in Eq. (17). As a result, the cell-edge in Soft FR archives higher coverage probability than in Strict FR, as shown in Figs. 4 and 5. For  $SNR = 15$  dB, and  $\beta = 5$ , the average rate the cell-edge user in Soft FR can archives 2.234 bit/Hz/s which is 60% larger than in Strict FR.



**Fig. 3** Variation of average coverage probability with different values of number of sub-bands

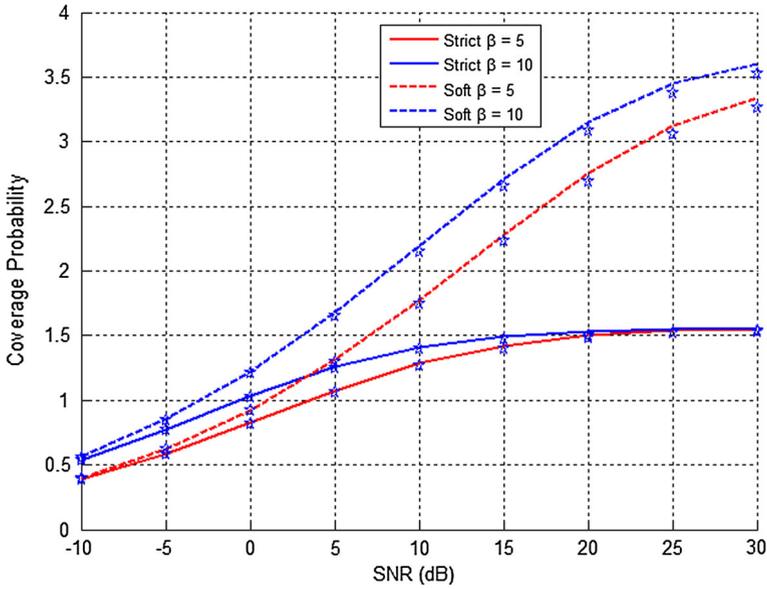


Fig. 4 Variation of average coverage probability of the cell-edge user with different values of  $\beta$

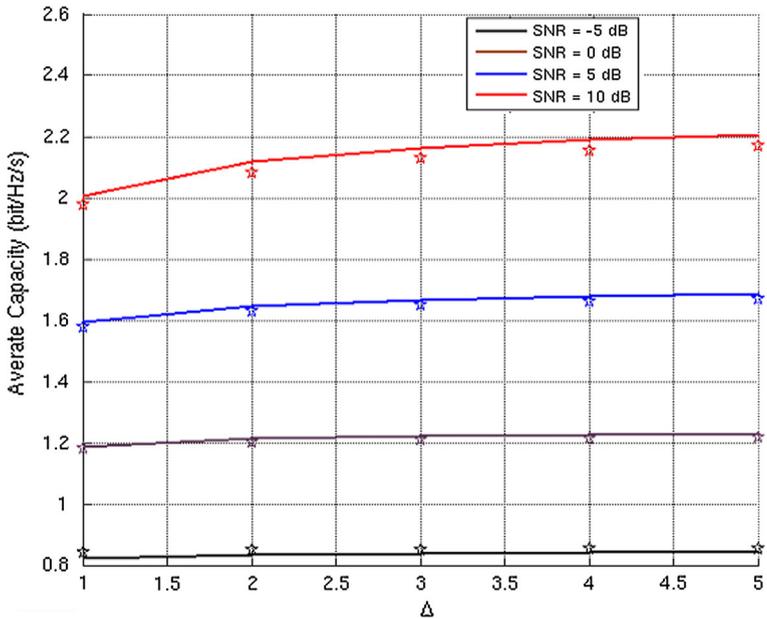


Fig. 5 Variation of average coverage probability of the cell-edge user in Soft FR with  $\beta = 10$

Figure 4 indicates that for Soft FR, an increase in SNR can significantly improve the average capacity of the cell-edge user, while for Strict FR, the average capacity slightly changes and rapidly reaches to the upper bound.

**Table 1** Increment of average coverage probability

$N$	$P_c$	$\Delta P_c(\%)$
2	0.3906	26.1137
6	0.4926	8.7900
15	0.5359	0.0097
20	0.5411	

It is observed from Fig. 5 that the average capacity of the cell edge-user slightly depends on  $\Delta$ . This is due to the fact that an increase in  $\Delta$  can reduce the number of base stations that use the same sub-band but this also increases the probability of usage the same sub-band at two base stations that use the same cell-edge sub-bands.

## 6 Conclusion

This paper presents a closed-form expression for average coverage probability of a typical user associated with the nearest BS in random cellular networks enabling frequency reuse in composite Rayleigh–Lognormal fading. The exact mathematical analysis for both low SNR and high SNR based on two well-known approximation rules, called Gauss–Hermite and Gauss–Legendre quadrature shows that Soft FR outperforms Strict FR in terms of average coverage probability and average capacity. In Strict FR, the results indicate that both average coverage probability and capacity do not depend on the frequency reuse factor. In Soft FR, the effective interference power factor is defined to estimate the average intercell interference that impacts on a typical user. From coverage probability point of view, Strict FR is considered as a special case of Soft FR.

## Appendix 1

The average coverage probability of the user on the cell-center sub-band with transmit power  $\zeta P$  is given by

$$\mathbb{P}(\text{SINR} > T_c) = \mathbb{P}\left(\frac{\zeta P g r^{-\alpha}}{I + \sigma^2} > T_c\right) \quad (24)$$

$$\begin{aligned} &= \mathbb{E}\left(\sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \exp\left(-\frac{T_c r^\alpha (I + \sigma^2)}{\zeta P \gamma(a_n)}\right)\right) \\ &= \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \exp(-f(n)\sigma^2) \mathbb{E}(\exp(-f(n)I)) \end{aligned} \quad (25)$$

in which  $\frac{T_c r^\alpha}{\zeta P \gamma(a_n)} = f(n)$

Considering the expectation

$$\begin{aligned} \mathbb{E}(\exp(-f(n)I)) &= \mathbb{E}\left[\exp\left(-f(n) \sum_{u \in \theta} \delta(i=j) \rho P_u g_u r_u^{-\alpha}\right)\right] \\ &= \mathbb{E}_\theta \left[\prod_{u \in \theta} \mathbb{E}_{g_u}(\exp(-f(n)\delta(i=j)\rho P_u g_u r_u^{-\alpha}))\right] \end{aligned}$$

Since  $g_u$  is Rayleigh–Lognormal fading channel then

$$= \mathbb{E}_\theta \left[ \prod_{u \in \theta} \sum_{n_1=1}^{N_p} \frac{w_{n_1}}{\sqrt{\pi}} \frac{1}{N} \frac{1}{1 + \gamma(a_{n_1})f(n)\rho P_u r_u^{-\alpha}} \right]$$

Using the properties of PPP probability generating function [10]

$$= \exp \left( -\pi\lambda \left( \sum_{n_1=1}^{N_p} \frac{w_{n_1}}{\sqrt{\pi}} \int_r^\infty 2 \left[ 1 - \frac{1}{1 + \gamma(a_{n_1})f(n)Pr_u^{-\alpha}} \right] r_u dr_u \right) \right)$$

Letting  $C = T_c \frac{\rho \gamma(a_{n_1})}{\gamma(a_n)}$ , using properties of Gamma function and Gauss–Legendre rule, the expectation can be approximated by [13]

$$= \exp \left( \frac{\pi\lambda r^2}{N} \sum_{n_1=1}^{N_p} \frac{w_{n_1}}{\sqrt{\pi}} \left( \frac{2}{\alpha} C^{\frac{2}{\alpha}} \frac{\pi}{\sin\left(\frac{\pi(\alpha-2)}{\alpha}\right)} - \sum_{n_{GL}=1}^{N_{GL}} \frac{c_i}{2} \frac{C}{C + \left(\frac{x_i+1}{2}\right)^{\alpha/2}} \right) \right) \tag{26}$$

$$= \exp \left( \frac{\pi\lambda r^2}{N} f_{SFR}(T_c, \zeta, \rho, n) \right)$$

## Appendix 2

The coverage probability expression of the cell-edge user is written in form of conditional probability.

$$P_{SofE}(T, \lambda, \alpha) = \frac{\mathbb{P} \left( \frac{\beta P g' r^{-\alpha}}{\sigma^2 + \eta l'_0} > T_c \mid \frac{P g r^{-\alpha}}{\sigma^2 + \eta l_0} < T \right)}{\mathbb{P} \left( \frac{P g r^{-\alpha}}{\sigma^2 + \eta l_0} < T \right)}$$

$$= \frac{\int_0^\infty 2\pi\lambda e^{-\pi\lambda r^2} \mathbb{P} \left( \frac{\beta P g' r^{-\alpha}}{\sigma^2 + \eta l'_0} > T_c, \frac{P g r^{-\alpha}}{\sigma^2 + \eta l_0} < T \right)}{\int_0^\infty 2\pi\lambda e^{-\pi\lambda r^2} \mathbb{P} \left( \frac{P g r^{-\alpha}}{\sigma^2 + \eta l_0} < T \right)}$$

Due to independent channel gain and independent interference, the average coverage probability equals

$$= \frac{\int_0^\infty 2\pi\lambda e^{-\pi\lambda r^2} \mathbb{P} \left( \frac{\beta P g' r^{-\alpha}}{\sigma^2 + \eta l'_0} > T_c \right) \mathbb{P} \left( \frac{P g r^{-\alpha}}{\sigma^2 + \eta l_0} < T \right)}{\int_0^\infty 2\pi\lambda e^{-\pi\lambda r^2} \mathbb{P} \left( \frac{P g r^{-\alpha}}{\sigma^2 + \eta l_0} < T \right)}$$

$$= \frac{\int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_c(T_c, N_E, \beta, \eta, r) P_0(T, N_c, 1, \eta, r) dt}{\int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_0(T, N_c, 1, \eta, r) dt}$$

Using Gauss–Legendre by letting  $r = \frac{t}{\sqrt{t}}$  and combining with Eqs. (8–10), we obtain the desired result.

## Appendix 3

In Appendix 2, the coverage probability is presented in an integral with variable  $r$

$$P_{\text{SofE}}(T, \lambda, \alpha) = \frac{\int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_c(T_c, N_E, \beta, \eta, r) P_0(T, N_c, 1, \eta, r) dt}{\int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_0(T, N_c, 1, \eta, r) dt}$$

Considering the denominator under condition  $\delta^2 = 0$ , we have

$$\begin{aligned} \int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_0(T, N_c, 1, \eta, r) dt &= \int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} \left[ 1 - \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} e^{-\frac{\pi\lambda r^2}{N_c} f_{\text{SFR}}(T, N_c, n, 1, \eta)} \right] dt \\ &= 1 - \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \int_0^\infty e^{-t} \left[ e^{-\frac{t}{N_c} f_{\text{SFR}}(T, N_c, n, 1, \eta)} \right] dt \\ &= 1 - \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \frac{1}{1 + \frac{1}{N_c} f_{\text{SFR}}(T, N_c, n, 1, \eta)} \end{aligned}$$

Considering the numerator, we obtain

$$\begin{aligned} &\int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_c(T_c, N_E, \beta, \eta, r) P_0(T, N_c, 1, \eta, r) dt \\ &= \int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_c(T_c, N_E, \beta, \eta, r) dt \\ &\quad - \int_0^\infty 2\pi\lambda r e^{-\pi\lambda r^2} P_c(T_c, N_E, \beta, \eta, r) [P_c(T, N_c, 1, \eta, r)] dt \\ &= \sum_{n=1}^{N_p} \frac{w_n}{\sqrt{\pi}} \frac{1}{1 + \frac{1}{N_E} f_{\text{SFR}}(T_c, n, \beta, \eta)} - \sum_{n=1}^{N_p} \sum_{m=1}^{N_p} \frac{w_m w_n}{\pi} \frac{1}{\frac{1}{N_E} f_{\text{SFR}}(T_c, \beta, \eta, n) + \frac{1}{N_c} f_{\text{SFR}}(T, 1, \eta, m)} \end{aligned}$$

The Lemma 3 is proved.

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