Inconsistency Measures for Probabilistic Knowledge Bases

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***Abstract*—One of the major concerns in knowledge integration is inconsistencies in knowledge bases. An inconsistency measure is a tool that helps analyzing inconsistency knowledge bases and resolving inconsistencies. In recent years, a wide range of measures with desirable properties have been proposed, however, these measures often correspond to logical, or probabilistic-logical framework. In this paper, we investigate several inconsistency measures and their properties for the knowledge bases represented by probabilistic framework.**

I. Introduction

Nowdays, many researches on artificial intelligence deal with inconsistency of the knowledge base. Inconsistencies may easily arise in many applications during the process of merging information from different source. For example, when some experts may share their knowledge to solve a problem. Because of their different perspectives, they may yield a common inconsistent knowledge base. Although these inconsistencies often affect only a small portion of the joint knowledge base or occur from only small differences in their knowledge, they cause severe damage. Therefore, solving inconsistency of knowledge is a basic and very essential subtask in many tasks of knowledge management [1]. Furthermore, the degree of inconsistency could affect the quality of collective knowledge [2]. Some approaches have been developed to resolve inconsistencies by adjusting conditional probabilities [3] or removing parts of the knowledge base [3], employing inconsistency measures to assess the extent of inconsistency [4-9]. An inconsistency measure assigns a non-negative real value to a knowledge base with the intended meaning that the larger the values the larger the inconsistency in knowledge base, with the value equals zero meaning that knowledge base is consistent.

Inconsistency measures for logic framework

A overview of measures for classical logics has been proposed in [10-12]. The idea of inconsistency measures is to based on formulae, variables, and distances, and to use Shapley Value. The class of those measures takes into account the number of formulae, the proportion of the propositional variables required to produce an inconsistency. Another approach is that measures rely on minimal inconsistent sets correspond to combinatorial optimization problems [11,13]. The opposite method to ones in [13] is to ultilize the set of maximal consistent subsets to define inconsistency measure [14].

Inconsistency measures for probabilistic-logical framework

A class of inconsistency measures for probabilistic logics can be found in [7]. The measures can be implemented by enumerating the minimal inconsistent sets, or by combining the numbers of minimal inconsistent sets that contain certain rules. In [6], two steps are proposed for implementing an inconsistency measure. The first step is to arrange the size of probability intervals in a vector. The second step is to measure the size of this vector with respect to a p-norm. The family of minimal violation measures that correspond to convex optimization problems computed by means of convex optimization techniques [15]. The class of those measures derives from the idea of measuring violation with respect to some norm like the euclidean. If the violation value is zero, the knowledge base is consistent. The class of the minimal violation measures with integrity constraints in [16] is extended from [15, 17].

Inconsistency measures for probabilistic framework

In [4, 18], measuring inconsistency in conditional probabilistic knowledge bases has focused on the practical aspects of efficiently resolving inconsistencies when merging probabilistic rule sets. In [3, 19], a set of heuristics that are used to restore consistency in a knowledge base not based on a theoretical elaboration. However, this result has been applied successfully to improve fraud detection in management. Several inconsistency measures for probabilistic framework can be found in [5]. Minimally changing the inconsistency knowledge base to a consistent one is the major idea of this class of measures. The unnormalized inconsistency measure is the solution of the optimization problem and Shapley inconsistency measure is defined by Shapley value that proposed in [20].

The inconsistency of knowledge base could bring about some conflicts between the knowledge states of a collective. The measure of the quality of collective knowledge depend on the distance from the collective knowledge to the real knowledge state. The quality of collective knowledge tends to be better if added members are closer to the real knowledge state. This approach has been put forward in [21].

The main contribution of this paper is twofold. First, we make a deeply survey on the classically inconsistency measures and adjust some of them to the probabilistic framework. Second, we investigate a set of desirable properties for inconsistency measures and point out the logical relation among them.

This paper is organized as follows. In Section II we start with some necessary preliminaries about the probability, the conditional probability, conditional probabilistic rules, probabilistic constraint, and knowledge bases represented by probabilistic framework. Afterwards, in Section III we investigate some desirable properties of inconsistency measures on probabilistic knowledge bases. In section IV we provide the definitions of inconsistency measures on probabilistic knowledge bases and the propositions of desirable properties investigated in Section III corresponds to each inconsistency measure which should be fulfilled. Conclusion and future work are in Section V.

II. Preliminaries

In this section we introduce two basic notions used in this work, namely a probability function and a probabilistic knowledge base. A knowledge base is a collection of entities, facts, relationships that conforms with a certain data model. To represent a knowledge base using probabilistic framework, the probability theory plays an important role. We will consider some of the concepts and principles of conditional probability that are very useful for knowledge representation.

A. Events and Probability

The set of all possible outcomes of a statistical experiment is called *the sample space* and is represented by the symbol S. Each outcome in a sample space is called a sample point. *An event* E is a subset of a sample space S. That is, an event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in E, then we say that E has occurred. Let $\mathsf{E} = \{E_1, ..., E_n\}$ be the set of events. For example, if the outcome of an experiment consists in the determination of the sex of a newborn child, then $\mathsf{S} = \{\mathrm{g}, \mathrm{b}\}$ where the outcome g means that the child is a girl and b that it is a boy; if $E = \{\mathrm{g}\}$, then E is the event that the child is a girl. Similarly, if $E = \{\mathrm{b}\}$, then E is the event that the child is a boy.
For $F, G \in \mathsf{E}$, conjunction $F \wedge G$ is abbreviated by FG and negation $\neg F$ by $\overline{F}$. The intersection of two events F, G, denoted by FG, is the event containing all elements that are common to F and G.
Let $\Gamma(\mathsf{E})$ be the set of all complete conjunctions of E, defined as:

$$\Gamma(\mathsf{E}) = \bigcup_{j=1}^{2^n} \{\Delta_j | \Delta_j = \tilde{E}_1 ... \tilde{E}_n, \quad \tilde{E}_i = E_i \text{ or } \tilde{E}_i = \bar{E}_i, 1 \le i \le n\}$$

The probability of an event E is the sum of the weights of all sample points in E, denoted by $\mathcal{P}(E)$. Therefore,

$$0 \le \mathcal{P}(\mathrm{E}) \le 1 \quad \text{and} \quad \mathcal{P}(\mathsf{S}) = 1$$

For any sequence of mutually exclusive events $E_1, E_2, ..., E_n$ (that is, events for which $E_i E_j = \emptyset$ when $i \ne j$),

$$\mathcal{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathcal{P}(E_i), \mathrm{n} = 1, 2, ..., \infty$$

The conditional probability of F, given G, denoted by $\mathcal{P}(F|G)$, is defined by

$$\mathcal{P}(F|G) = \frac{\mathcal{P}(FG)}{\mathcal{P}(G)} = \rho, \; provided \; \mathcal{P}(G) > 0$$

Two events F and G are *independent* if and only if $\mathcal{P}(F|G) = \mathcal{P}(F)$ *or* $\mathcal{P}(G|F) = \mathcal{P}(G)$ assuming the existences of the conditional probabilities. Otherwise, F and G are *dependent*.
Conditional probabilistic rules:

- $\mathcal{P}(FG) = \mathcal{P}(F)\mathcal{P}(G|F) = \mathcal{P}(G)\mathcal{P}(F|G)$ provided $\mathcal{P}(F) > 0$, $\mathcal{P}(G) > 0$
- $P(G_k|F) = \frac{P(G_k)P(F|G_k)}{\sum_{i=1}^{n} P(G_i)P(F|G_i)} \forall k = 1, ..., n$
- $\mathcal{P}(F) = \sum_{i=1}^{n} \mathcal{P}(G_i)\mathcal{P}(F|G_i)$

B. Probabilistic knowledge base

A probabilistic constraint κ is an expression of the form $(F|G)[\rho]$ with $F, G \in \mathsf{E}$ and $0 \le \rho \le 1$ over S. Intuitively, a constraint $(F|G)[\rho]$ that our knowledge in F given that G hold is ρ. If G is tautological, $G \equiv \top$, we abbreviate $(F|\top)[\rho]$ by $(F)[\rho]$.

Definition 1. (Probabilistic knowledge base). *Probabilistic knowledge base* $\mathcal{K}$ is a finite set of probabilistic constraints defined as:

$$\mathcal{K} = \langle \kappa_1, ..., \kappa_n \rangle$$

where $\kappa_i = (F_i|G_i)[\rho_i], \forall i = 1, ..., n$

Let $\mathbb{K}$ be the set of all knowledge bases. Let $\mathbb{R}_{\ge 0}$ be the set of non-negative real values including $+\infty$. Let $\mathbb{R}_{[0,1]}$ be the set of all real values from 0 to 1.
A complete conjunction $\Delta_j \in \Gamma(\mathsf{E})$ for $j = 1, ..., 2^n$ satisfies an event F, denoted by $\Delta_j \models F$ if and only if F positively appears in Δ_j.
Let $\mathsf{SE}(H) = \{\Delta_j \in \Gamma(\mathsf{E}) | \Delta_j \models H\}$ for $j = 1, ..., 2^n$, where H is an event or a set of events.
Let $\mathcal{P}_{\mathcal{K}} : \Gamma(\mathsf{E}) \to \mathbb{R}_{[0,1]}$ be a probability function that fulfills all restrictions on the conditional probabilities imposed by the probabilistic constraints in $\mathcal{K}$.
Let $\Upsilon(\mathsf{E})$ be the set of all probability functions $\mathcal{P}_{\mathcal{K}}$, defined by $\Upsilon(\mathsf{E}) = \bigcup_{j=1}^{2^n} \{\mathcal{P}_{\mathcal{K}}(\Delta_j)\}$.
A probability function $\mathcal{P}_{\mathcal{K}} \in \Upsilon(\mathsf{E})$ satisfies a probabilistic constraint $(F|G)[\rho]$, denoted by $\mathcal{P}_{\mathcal{K}} \models (F|G)[\rho]$, if and only if $\mathcal{P}_{\mathcal{K}}(FG) = \rho\mathcal{P}_{\mathcal{K}}(G)$.
A probability function $\mathcal{P}_{\mathcal{K}}$ satisfies $\mathcal{K}$, denoted by $\mathcal{P}_{\mathcal{K}} \models \mathcal{K}$, if and only if $\mathcal{P}_{\mathcal{K}} \models \kappa, \;\; \forall \kappa \in \mathcal{K}$.
With $F \in \mathsf{E}$,

$$\mathcal{P}_{\mathcal{K}}(F) = \sum_{\Delta_j \in \Gamma(\mathsf{E}), \Delta_j \models F (j=1,...2^n)} \mathcal{P}_{\mathcal{K}}(\Delta_j)$$

For every constraint $(F_i|G_i)[\rho_i]$, $i = 1, .., n$, we write:

$$\sum_{\Delta_j \in SE(F_i G_i)} \mathcal{P}_{\mathcal{K}}(\Delta_j) = (\rho_i + x_i - y_i) \, . \sum_{\Delta_j \in SE(G_i)} \mathcal{P}_{\mathcal{K}}(\Delta_j)$$

$\forall j = 1, ..., 2^n$, where $x_i, y_i\,(\forall i = 1, .., n)$ are deviations to avoid determining absolute values when summing the deviations.

Let $\tilde{\mathcal{P}} : \Upsilon(\mathsf{E}) \to \mathbb{R}_{[0,1]}$ be a probability function in $\Upsilon(\mathsf{E})$ such that $\tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}}) > 0$ only for finitely many $\mathcal{P}_{\mathcal{K}} \in \Upsilon(\mathsf{E})$.

Let $\tilde{\Gamma}(\mathsf{E})$ be the set of all probability functions $\tilde{\mathcal{P}}$, defined by $\tilde{\Gamma}(\mathsf{E}) = \{\tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}}) | \tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}}) > 0, \mathcal{P}_{\mathcal{K}} \in \Upsilon(\mathsf{E})\}$.

Let $\tilde{\mathcal{P}}(\kappa)$ be a probability of constraint κ that satisfies κ, defined as:

$$\tilde{\mathcal{P}}(\kappa) = \sum_{\mathcal{P}_{\mathcal{K}} \in \Upsilon(\mathsf{E}), \mathcal{P}_{\mathcal{K}} \models \kappa} \tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}})$$

Example 1. Vietnam Airlines make a survey about their flights. They get the opinions of customers, attendants and pilots:

- Customers think the probability that a regularly scheduled flight departs on time (D) is ρ_1; the probability that it arrives on time (A) is ρ_2; and the probability that a plane arrives on time when it departed on time is ρ_3.
- Stewards and Pilots say the probability that a regularly scheduled flight departs on time (D) is ρ_4; the probability that it arrives on time (A) is ρ_5; and the probability that a plan departed on time when it has arrived on time is ρ_6.

We have probabilistic knowledge bases as follows:

$\mathcal{K}_1 = \langle (D)[\rho_1], (A)[\rho_2], (A|D)[\rho_3] \rangle$

$\mathcal{K}_2 = \langle (D)[\rho_4], (A)[\rho_5], (D|A)[\rho_6] \rangle$

III. Desirable properties for inconsistency measures

In knowledge-based systems the notion consistency of knowledge is most often understood as a situation in which a knowledge base does not contain contradictions. Thus inconsistency of knowledge appears if there are some contradictions. However, this definition is not satisfactory because the notion inconsistency is only replaced by the notion contradiction. Contradiction is easily defined in stood as a situation in which a set of logic formulae has no model; that is, on the basis of these formulae one can infer false. For example, for a set nonlogic knowledge bases the notion of contradiction is more difficult to define. The notion of inconsistency of knowledge is more complex than the notion of contradiction.

We are interested in determining for a specific knowledge base $\mathcal{K}$, whether $\mathcal{K}$ is consistent if there is at least one probability function $\mathcal{P}_{\mathcal{K}}$ with $\mathcal{P}_{\mathcal{K}} \models \mathcal{K}$ or inconsistent if there is no such $\mathcal{P}_{\mathcal{K}}$. Let $\mathsf{PS}(\mathcal{K})$ be the set of probability functions $\mathcal{P}_{\mathcal{K}}$ with $\mathcal{P}_{\mathcal{K}} \models \mathcal{K}$, defined by $\mathsf{PS}(\mathcal{K}) = \{\mathcal{P}_{\mathcal{K}} | \mathcal{P}_{\mathcal{K}} \models \mathcal{K}\}$. Therefore, if $\mathsf{PS}(\mathcal{K}) = \emptyset$ then $\mathcal{K}$ is inconsistent, denoted by $\mathcal{K} \models \perp$. Otherwise, $\mathcal{K}$ is consistent, denoted by $\mathcal{K} \not\models \perp$.

Example 2. Consider the knowledge bases from Example 1 with $\rho_1 = 0.7, \rho_2 = 0.5, \rho_3 = 0.9, \rho_4 = 0.4, \rho_5 = 0.75, \rho_6 = 0.85$, we have:

$\mathcal{K}_1 = \langle (D)[0.7], (A)[0.5], (A|D)[0.9] \rangle$

$\mathcal{K}_2 = \langle (D)[0.4], (A)[0.75], (D|A)[0.85] \rangle$

Detecting that $\mathsf{PS}(\mathcal{K}_1) = \emptyset$ as $\mathcal{P}_{\mathcal{K}_1} \models \langle (D)[0.7], (A|D)[0.9] \rangle$ results in $\mathcal{P}_{\mathcal{K}_1}(A) \geq 0.7 \times 0.9 = 0.63$ which can not concurrently be satisfied with $\mathcal{P}_{\mathcal{K}_1}(A) = 0.5$. Therefore, $\mathcal{K}_1 \models \perp$. Similarly, $\mathcal{K}_2 \models \perp$.

Definition 2. (Inconsistency measure). *Inconsistency measure* $\mathcal{IM}$ is a function $\mathcal{IM} : \mathbb{K} \to \mathbb{R}_{\geq 0}$ such that $\mathcal{IM}(\mathcal{K}) = 0$ if and only if $\mathsf{PS}(\mathcal{K}) \neq \emptyset$, $\mathcal{K} \in \mathbb{K}$.

A knowledge base is consistent if there is at least one interpretation that satisfies all its formula. If a knowledge base $\mathcal{K}$ is inconsistent, then one can define the minimal inconsistent subsets of $\mathcal{K}$ as:

Definition 3. (Minimal inconsistent subset). A set of probabilistic constraints $\mathcal{CMP} \subseteq \mathcal{K}$ is *minimal inconsistent subset* if $\mathcal{CMP} \models \perp$ and every $\mathcal{CMP}_s \subset \mathcal{CMP}$ then $\mathcal{CMP}_s \not\models \perp$.

Let $\mathsf{SMI}(\mathcal{K})$ be the set of the minimal inconsistent subsets of $\mathcal{K} \in \mathbb{K}$, defined as:

$$\mathsf{SMI}(\mathcal{K}) = \{\mathcal{CMP} \subseteq \mathcal{K} | \mathcal{CMP} \models \perp \\ and\ \forall \mathcal{CMP}_s \subset \mathcal{CMP} : \mathcal{CMP}_s \not\models \perp\}$$

Definition 4. (Maximal consistent subset). A set of probabilistic constraints $\mathcal{MCS} \subseteq \mathcal{K}$ is *maximal consistent subset* if $\mathcal{MCS} \not\models \perp$ and every $\mathcal{K} \supseteq \mathcal{MCS}_s \supset \mathcal{MCS}$ then $\mathcal{MCS}_s \models \perp$

Let $\mathsf{SMC}(\mathcal{K})$ be the set of the maximal consistent subsets of $\mathcal{K} \in \mathbb{K}$, defined as:

$$\mathsf{SMC}(\mathcal{K}) = \{\mathcal{MCS} \subseteq \mathcal{K} | \mathcal{MCS} \not\models \perp \\ and\ \forall \mathcal{K} \supseteq \mathcal{MCS}_s \supset \mathcal{MCS} : \mathcal{MCS}_s \models \perp\}$$

Let $\mathsf{SCC}(\mathcal{K})$ be the set of self-contradictory constraints of of $\mathcal{K} \in \mathbb{K}$, defined as:

$$\mathsf{SCC}(\mathcal{K}) = \{\kappa \in \mathcal{K} | \kappa \models \perp\}$$

Definition 5. (Free constraint). A probabilistic constraint $\kappa \in \mathcal{K}$ is *free constraint* in $\mathcal{K}$ if and only if $\kappa \notin \mathcal{CMP}$ for all $\mathcal{CMP} \in \mathsf{SMI}(\mathcal{K})$.

Let $\mathsf{Fc}(\mathcal{K})$ be the set of all free constraints of $\mathcal{K}$, defined as:

$$\mathsf{Fc}(\mathcal{K}) = \{\kappa \in \mathcal{K} | \kappa \notin \mathcal{CMP},\ \forall \mathcal{CMP} \in \mathsf{SMI}(\mathcal{K})\}$$

Let $\mathsf{App}(CoB)$ denote the set of atoms appearing in CoB where CoB is a constraint or a knowledge base.

Definition 6. (Safe constraint). A probabilistic constraint $\kappa \in \mathcal{K}$ is *safe constraint* in $\mathcal{K}$ if and only if $\mathsf{App}(\kappa) \cap \mathsf{App}(\mathcal{K} \backslash \{\kappa\}) = \emptyset$.

Let $\mathsf{Sc}(\mathcal{K})$ be the set of all safe constraints of $\mathcal{K}$, defined as:

$$\mathsf{Sc}(\mathcal{K}) = \{\kappa \in \mathcal{K} | \mathsf{App}(\kappa) \cap \mathsf{App}(\mathcal{K} \backslash \{\kappa\}) = \emptyset\}$$

In the rest later, we will put forward several properties of $\mathcal{IM}$ which are used to characterize inconsistency measure. Some of the following properties are adapted from (Hunter and Konieczny, 2006) and rewritten to fit a probabilistic framework.

Consistency (CON): If $\mathcal{K}$ is consistent, then $\mathcal{IM}(\mathcal{K}) = 0$.

CON describes the minimal requirement for an inconsistency measure $\mathcal{IM}(\mathcal{K})$

Inconsistency (ICO): If $\mathcal{K}$ is inconsistent, then $\mathcal{IM}(\mathcal{K}) > 0$
ICO requires that every inconsistent knowledge base has a strictly positive inconsistency value.
Monotonicity (MON): It holds that $\mathcal{IM}(\mathcal{K}) \leq \mathcal{IM}(\mathcal{K} \cup \kappa)$.
MON requires that $\mathcal{IM}$ will increase under adding constraints to the knowledge base.
Super-Additivity (SUA): If $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$, it is $\mathcal{IM}(\mathcal{K}_1 \cup \mathcal{K}_2) \geq \mathcal{IM}(\mathcal{K}_1) + \mathcal{IM}(\mathcal{K}_2)$
SUA requires that the inconsistency value of the comon knowledge base is not smaller the sum of the inconsistency values of two individual knowledge bases. Note that SUA is the stronger property, as it can be easily seen that SUA implies MON.
Normalization (NOR): It holds that $\mathcal{IM}(\mathcal{K}) \in [0, 1]$
NOR expresses that $\mathcal{IM}$ is always in the unit interval.
MI-separability (MIS): If $\mathsf{SMI}(\mathcal{K}_1 \cup \mathcal{K}_2) = \mathsf{SMI}(\mathcal{K}_1) \cup \mathsf{SMI}(\mathcal{K}_2)$ and $\mathsf{SMI}(\mathcal{K}_1) \cap \mathsf{SMI}(\mathcal{K}_2) = \emptyset$ then $\mathcal{IM}(\mathcal{K}_1 \cup \mathcal{K}_2) = \mathcal{IM}(\mathcal{K}_1) + \mathcal{IM}(\mathcal{K}_2)$
MIS demands that if the minimal consistent subsets of $\mathcal{K}_1$ and $\mathcal{K}_2$ are separated, the inconsistency value of uninion of two knowledge base $\mathcal{K}_1 \cup \mathcal{K}_2$ is the sum of the inconsistency values of $\mathcal{K}_1$ and $\mathcal{K}_2$.
Free-constraint independence (FCI): If $\kappa \in \mathsf{Fc}(\mathcal{K})$ then $\mathcal{IM}(\mathcal{K}) = \mathcal{IM}(\mathcal{K} \backslash \{\kappa\})$.
FCI expresses that the inconsistency value should not be changed when eliminating a free constraint from $\mathcal{K}$.
Safe-constraint independence (SCI): If $\kappa \in \mathsf{Sc}(\mathcal{K})$ then $\mathcal{IM}(\mathcal{K}) = \mathcal{IM}(\mathcal{K} \backslash \{\kappa\})$
SCI states that the inconsistency value should not be changed when eliminating a safe constraint from $\mathcal{K}$.
Penalty (PEN): If $\kappa \notin \mathsf{Fc}(\mathcal{K})$ and $\kappa \in \mathcal{K}$ then $\mathcal{IM}(\mathcal{K}) > \mathcal{IM}(\mathcal{K} \backslash \{\kappa\})$
PEN requires that adding inconsistent information increases the inconsistency value.

Definition 7. (Basic inconsistency measure). An inconsistency measure of knowledge base $\mathcal{K}$ is called *a basic inconsistency measure*, denoted $\mathcal{IM}_b(\mathcal{K})$, if it satisfies the following properties: CON, NOR, MON, FCI

IV. Inconsistency Measures

We consider existing approaches to inconsistency measurement for classical logic, probabilistic-logical and adjust those to the probabilistic framework.

Definition 8. (Drastic inconsistency measure). Function $\mathcal{IM}_{dr}(\mathcal{K})$ is called *a drastic inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_{dr} : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}_{dr}(\mathcal{K}) = \begin{cases} 1 & if\ \mathcal{K} \models \perp, \\ 0 & otherwise. \end{cases}$$

for $\mathcal{K} \in \mathbb{K}$.

Proposition 1. The function $\mathcal{IM}_{dr}$ fulfills CON, MON, NOR, FCI, and SCI.

It is easy to see that the function $\mathcal{IM}_{dr}$ violates SUA, MIS, and PEN.

Proposition 2. The function $\mathcal{IM}_{dr}$ is a basic inconsistency measure.

Example 3. Consider the knowledge bases $\mathcal{K}_1$ and $\mathcal{K}_2$ from Example 2. It follows that $\mathcal{IM}_{dr}(\mathcal{K}_1) = 1$ and similarly $\mathcal{IM}_{dr}(\mathcal{K}_2) = 1$. It follows that $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$ and $\mathcal{IM}_{dr}(\mathcal{K}_1 \cup \mathcal{K}_2) = 1 < \mathcal{IM}_{dr}(\mathcal{K}_1) + \mathcal{IM}_{dr}(\mathcal{K}_2) = 2$ thus failing to satisfy SUA.
It follows that $\mathsf{SMI}(\mathcal{K}_1) = \{(D)[0.7], (A)[0.5], (A|D)[0.9]\}$ and $\mathsf{SMI}(\mathcal{K}_2) = \{(D)[0.4], (A)[0.75], (A|D)[0.85]\}$ thus violating MIS.
Furthermore, observe that violating PEN as $\mathsf{Fc}(\mathcal{K}_1) = \emptyset$ and $\mathsf{Fc}(\mathcal{K}_2) = \emptyset$.

Definition 9. (MI-inconsistency measure). Function $\mathcal{IM}_{mi}(\mathcal{K})$ is called MI-*inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_{mi} : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}_{mi}(\mathcal{K}) = |\mathsf{SMI}(\mathcal{K})|$$

for $\mathcal{K} \in \mathbb{K}$.

Proposition 3. The function $\mathcal{IM}_{mi}$ fulfills CON, MON, SUA, MIS, FCI, SCI, and PEN.

It is easy to see that the function $\mathcal{IM}_{mi}$ satisfies all desirable properties except NOR.

Example 4. Consider $\mathcal{K}_1$ and $\mathcal{K}_2$ from Example 2. We have $\mathcal{IM}_{mi}(\mathcal{K}_1) = 1$ and $\mathcal{IM}_{mi}(\mathcal{K}_2) = 1$. Hence violating NOR, because $\mathcal{IM}_{mi}(\mathcal{K}_1 \cup \mathcal{K}_2) = 2 \notin [0, 1]$.

Definition 10. (SMI^C-inconsistency measure). Function $\mathcal{IM}^c_{smi}(\mathcal{K})$ is called SMI^C*-inconsistency measure* whenever it is defined as follows: $\mathcal{IM}^c_{smi} : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}^c_{smi}(\mathcal{K}) = \sum_{\mathcal{CMP} \in \mathsf{SMI}(\mathcal{K})} \frac{1}{|\mathcal{CMP}|}$$

for $\mathcal{K} \in \mathbb{K}$.

Note that if $\mathsf{SMI}(\mathcal{K}) = \emptyset$, $\mathcal{IM}^c_{smi}(\mathcal{K}) = 0$.

Proposition 4. The function $\mathcal{IM}^c_{smi}$ fulfills CON, MON, SUA, MIS, FCI, SCI, and PEN.

It is easy to see that the function $\mathcal{IM}^c_{smi}$ only violates NOR.

Example 5. Consider $\mathcal{K}_1$ and $\mathcal{K}_2$ from Example 2, it follows that $\mathcal{IM}^c_{smi}(\mathcal{K}_1) \approx 0.33$ and $\mathcal{IM}^c_{smi}(\mathcal{K}_2) \approx 0.33$.

Definition 11. ($\ell-$inconsistency measure). Function $\mathcal{IM}_\ell(\mathcal{K})$ is called $\ell-$*inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_\ell : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}_\ell(\mathcal{K}) = 1 - \max\{\ell | \exists \tilde{\mathcal{P}} \in \tilde{\Gamma}(\mathsf{E}) : \forall \kappa \in \mathcal{K} : \tilde{\mathcal{P}}(\kappa) \geq \ell\}$$

for $\mathcal{K} \in \mathbb{K}$.

Proposition 5. The function $\mathcal{IM}_\ell$ fulfills CON, MON, NOR, FCI, SCI.

It is easy to see that the function $\mathcal{IM}_\ell$ satisfies neither SUA, MIS, nor PEN.

Proposition 6. The function $\mathcal{IM}_\ell$ is a basic inconsistency measure.

Example 6. Consider $\mathcal{K}_1$ from Example 2,
Let $\mathcal{K}_{11} = \{(D)[0.7], (A)[0.5]\}$
Let $\mathcal{K}_{12} = \{(D)[0.7], (A|D)[0.9]\}$
Let $\mathcal{K}_{13} = \{(A)[0.5], (A|D)[0.9]\}$
it follows that: $\mathcal{P}_{\mathcal{K}_{11}} \models \mathcal{K}_{11}$, $\mathcal{P}_{\mathcal{K}_{12}} \models \mathcal{K}_{12}$, $\mathcal{P}_{\mathcal{K}_{13}} \models \mathcal{K}_{13}$.
We have $\tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}_{11}}) = \tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}_{12}}) = \tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}_{13}}) \approx 0.33$, and $\tilde{\mathcal{P}}(\mathcal{P}_{\mathcal{K}}) = 0$ for $\mathcal{P}_{\mathcal{K}} \in \Upsilon(\{A, D\}) \backslash \{\mathcal{P}_{\mathcal{K}_{11}}, \mathcal{P}_{\mathcal{K}_{12}}, \mathcal{P}_{\mathcal{K}_{13}}\}$. Hence, we have $\tilde{\mathcal{P}}((D)[0.7])) = \tilde{\mathcal{P}}((A)[0.5])) = \tilde{\mathcal{P}}((A|D)[0.9])) \approx 0.66$.
Therefore, it follows $\mathcal{IM}_\ell(\mathcal{K}_1) = 1 - \frac{2}{3} \approx 0.33$ and similarly $\mathcal{IM}_\ell(\mathcal{K}_2) \approx 0.33$. However, $\mathcal{IM}_\ell(\mathcal{K}_1 \cup \mathcal{K}_2) \approx 0.33 < \mathcal{IM}_\ell(\mathcal{K}_1) + \mathcal{IM}_\ell(\mathcal{K}_2) \approx 0.66$ thus failing to satisfy both SUA, MIS. Similarly in Example 3, $\mathcal{IM}_\ell(\mathcal{K}_1)$ and $\mathcal{IM}_\ell(\mathcal{K}_2)$ violate PEN.

Definition 12. ($\chi-$inconsistency Measure). Function $\mathcal{IM}_\chi(\mathcal{K})$ is called $\chi-$*inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_\chi : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}_\chi(\mathcal{K}) = |\mathsf{SMC}| + |\mathsf{SCC}| - 1$$

for $\mathcal{K} \in \mathbb{K}$.

Proposition 7. The function $\mathcal{IM}_\chi$ fulfills CON, MON, SUA, MIS, FCI, SCI, and PEN.

It is easy to see that the function $\mathcal{IM}_\chi$ satisfies all desirable properties except NOR.

Example 7. Consider $\mathcal{K}_1$ and $\mathcal{K}_2$ from Example 2. Observe that SMC $= \{\mathcal{K}_{11}, \mathcal{K}_{121}, \mathcal{K}_{13}\}$ and SCC $= \emptyset$ implies $\mathcal{IM}_\chi(\mathcal{K}_1) = 3 + 0 - 1 = 2$ and similarly $\mathcal{IM}_\chi(\mathcal{K}_2) = 2$, therefore violating NOR.

Definition 13. ($\mu-$inconsistency measure). Function $\mathcal{IM}_\mu(\mathcal{K})$ is called $\mu-$*inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_\mu : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}_\mu(\mathcal{K}) = |\mathcal{K}| - max\{\xi | \forall \mathcal{MCS} \subseteq K : |\mathcal{MCS}| = \xi : \mathcal{MCS} \not\models \bot\}$$

for $\mathcal{K} \in \mathbb{K}$.

Proposition 8. The function $\mathcal{IM}_\mu$ fulfills CON, MON, SUA, MIS, FCI, SCI, and PEN.

It is easy to see that the function $\mathcal{IM}_\mu$ perform well with respect to all properties expect NOR.

Example 8. Consider $\mathcal{K}_1$ and $\mathcal{K}_2$ from Example 2. Observe that $|\mathcal{K}_1| = 3$ and $max\{|\mathcal{K}_{11}|, |\mathcal{K}_{12}|, |\mathcal{K}_{13}|\} = 2$ implies $\mathcal{IM}_\mu(\mathcal{K}_1) = 3 - 2 = 1$ and similarly $\mathcal{IM}_\mu(\mathcal{K}_2) = 1$.

Definition 14. (DM-inconsistency measure). Function $\mathcal{IM}^p_{dm}$ is called the DM-*inconsistency measure* whenever it is defined as follows: $\mathcal{IM}^p_{dm} : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}^p_{dm}(\mathcal{K}) = inf\{d^p_n(\vec{x}, \vec{y}) | \mathcal{K} = \mathcal{K}[\vec{x}] \ and \ \mathcal{K}[\vec{y}] \not\models \bot\}$$

for $\mathcal{K} \in \mathbb{K}$, where:

$$d^p_n(\vec{x}, \vec{y}) = \sqrt[p]{|x_1 - y_1|^p + ... + |x_n - y_n|^p}$$

for $\vec{x} = \langle x_1, ..., x_n\rangle$, $\vec{y} = \langle y_1, ..., y_n\rangle \in \mathbb{R}^n$ and $n = |\mathcal{K}|$.

Proposition 9. The function $\mathcal{IM}^p_{dm}$ fulfills CON, MON, FCI, and SCI.

Proposition 10. If p=1 then the function $\mathcal{IM}^p_{dm}$ fulfills SUA and MIS.

It is easy to see that $\mathcal{IM}^p_{dm}$ satisfies neither NOR, nor PEN.

Example 9. Consider $\mathcal{K}_1$, $\mathcal{K}_2$ from Example 2. We have $\vec{x} = \langle 0.7, 0.5, 0.9\rangle$ corresponding to $\mathcal{K}_1$. We look for a consistent knowledge base $\mathcal{K}_1[\vec{y}] = \langle (D)[0.5], (A)[0.5], (A|D)[1]\rangle$. Here, $\vec{y} = \langle 0.5, 0.5, 1\rangle$. It is easy to see that $\mathcal{IM}^p_{dm}(\mathcal{K}_1) = \sqrt[p]{0.2^p + 0.1^p}$. Therefore, with $p = 1$ and $p = 2$ it follows that $\mathcal{IM}^1_{dm}(\mathcal{K}_1) = 0.3$ respectively, and $\mathcal{IM}^2_{dm}(\mathcal{K}_1) \approx 0,224$ and similarly $\mathcal{IM}^1_{dm}(\mathcal{K}_2) = 0.5$ and $\mathcal{IM}^2_{dm}(\mathcal{K}_2) \approx 0,31$.
For $p = 1$ it follows that $\mathcal{IM}^1_{dm}(\mathcal{K}_1) + \mathcal{IM}^1_{dm}(\mathcal{K}_2) = \mathcal{IM}^1_{dm}(\mathcal{K}_1 \cup \mathcal{K}_2) = 0.8$, therefore satisfying both MIS and SUA
However, for $p = 2$ it follows that $\mathcal{IM}^2_{dm}(\mathcal{K}_1) + \mathcal{IM}^2_{dm}(\mathcal{K}_2) \approx 0.534 > \mathcal{IM}^2_{dm}(\mathcal{K}_1 \cup \mathcal{K}_2) \approx 0.381$, therefore failing to satisfy both MIS and SUA.
Similarly in Example 3, $\mathcal{IM}^p_{dm}(\mathcal{K}_1)$ and $\mathcal{IM}^p_{dm}(\mathcal{K}_2)$ violate PEN.

Example 10. Consider the knowledge bases from Example 1 with $\rho_1 = 1$, $\rho_2 = 0$, $\rho_3 = 0.3$:
$\mathcal{K}_1 = \langle (D)[1.0], (A)[0.0], (A|D)[0.3]\rangle$
We have $\vec{x} = \langle 1, 0, 0.3\rangle$. We look for a consistent knowledge base $\mathcal{K}_1[\vec{y}] = \langle (D)[0.5], (A)[0.5], (A|D)[0.5]\rangle$. Therefore, with $p = 1$ it follows that $\mathcal{IM}^1_{dm}(\mathcal{K}_1) = 1.2$ violates NOR .

Definition 15. (SUM-inconsistency measure). Function $\mathcal{IM}_{sum}(\mathcal{K})$ is called SUM-*inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_{sum} : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}_{sum}(\mathcal{K}) = \sum_{\mathcal{CMP} \in \mathsf{SMI}(\mathcal{K})} \mathcal{IM}^p_{dm}(\mathcal{CMP})$$

for $\mathcal{K} \in \mathbb{K}$.

Proposition 11. The function $\mathcal{IM}_{sum}$ fulfills CON, MON, SUA, MIS, FCI, SCI, and PEN.

It is easy to see that the function $\mathcal{IM}_{sum}$ only violates NOR.

Example 11. Consider $\mathcal{K}_1$ from Example 9, it follows that $\mathcal{IM}_{sum}(\mathcal{K}_1) = 1.2$ thus failing to satisfy NOR.

Definition 16. (Probabilistic Shapley Inconsistency Value). Let $\mathcal{K}$ be a knowledge base, Probabilistic Shapley Inconsistency Value of $\kappa \in \mathcal{K}$, noted $\mathsf{SIV}_\mathcal{K}(\kappa)$, defined as:

$$\mathsf{SIV}_\mathcal{K}(\kappa) = \sum_{\mathcal{K}_s \subseteq \mathcal{K}} \frac{(m-1)!(n-m)!}{n!} d$$

where $m = |\mathcal{K}_s|$, $n = |\mathcal{K}|$, $d = \mathcal{IM}_b(\mathcal{K}_s) - \mathcal{IM}_b(\mathcal{K}_s \backslash \{\kappa\})$

Definition 17. (SV-inconsistency measure). Function $\mathcal{IM}_{sv}(\mathcal{K})$ is called SV$-$*inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_{sv} : \mathbb{K} \to \mathbb{R}_{\geq 0}$,

$$\mathcal{IM}_{sv}(\mathcal{K}) = \max_{\kappa \in \mathcal{K}} \{\mathsf{SIV}_\mathcal{K}(\kappa)\}$$

for $\mathcal{K} \in \mathbb{K}$.

Proposition 12. The function $\mathcal{IM}_{sv}$ fulfills CON, FCI, NOR.

Example 12. Consider $\mathcal{K}_1$ and $\mathcal{K}_2$ from Example 2 with $\mathcal{IM}_{dr}(\mathcal{K}_1) = 1$, $\mathcal{IM}_{dr}(\mathcal{K}_2) = 1$ which are basic inconsistency measure. There it is $\mathsf{SIV}_{\mathcal{K}_1}((D)[0.7]) \approx 0.33$, $\mathsf{SIV}_{\mathcal{K}_1}((A)[0.5]) \approx 0.33$, $\mathsf{SIV}_{\mathcal{K}_1}((A|D)[0.9]) \approx 0.33$. So $\mathcal{IM}_{sv}(\mathcal{K}_1) \approx \max\{0.33, 0.33, 0.33\} \approx 0.33$, similarly $\mathcal{IM}_{sv}(\mathcal{K}_2) = 0.33$.

Definition 18. (Unnormalized inconsistency measure). Function $\mathcal{IM}_{umi}(\mathcal{K})$ is called *unnormalized inconsistency measure* whenever it is defined as follows: $\mathcal{IM}_{umi} : \mathbb{K} \to \mathbb{R}_{\geq 0}$, $\mathcal{IM}_{umi}(\mathcal{K})$ is the solution of the optimization problem:

$$\min\ f(x_1, ..., x_n, y_1, ..., y_n) = x_1 + ... + x_n + y_1 + ... + y_n \tag{1}$$

subject to

$$\sum_{\Delta_j \in SE(F_i G_i)} \mathcal{P}_{\mathcal{K}}(\Delta_j) = (\rho_i + x_i - y_i) \, . \sum_{\Delta_j \in SE(G_i)} \mathcal{P}_{\mathcal{K}}(\Delta_j) \tag{2}$$

$\forall\, j = 1, ..., 2^n$

$$0 \leq \rho_1 + x_1 - y_1 \leq 1, ..., 0 \leq \rho_n + x_n - y_n \leq 1 \tag{3}$$

$$\sum_{j=1}^{2^n} \mathcal{P}_{\mathcal{K}}(\Delta_j) = 1 \tag{4}$$

$$\forall 0 \leq j \leq 2^n, \mathcal{P}_{\mathcal{K}}(\Delta_j) \geq 0 \tag{5}$$

Proposition 13. The function $\mathcal{IM}_{umi}$ fulfills CON, MON, SUA, FCI, SCI, and PEN.

Example 13. Consider $\mathcal{K}_1$ and $\mathcal{K}_2$ from Example 2. We have optimization problem with the following constraints:

$$\min\ f(x_1, ..., x_n, y_1, ..., y_n) = x_1 + ... + x_n + y_1 + ... + y_n \tag{6}$$

subject to

$$0 \leq 0.7 + x_1 - y_1 \leq 1, 0 \leq 0.5 + x_2 - y_2 \leq 1, 0 \leq 0.9 + x_3 - y_3 \leq 1 \tag{7}$$

$$\mathcal{P}_{\mathcal{K}}(DA) + \mathcal{P}_{\mathcal{K}}(D\bar{A}) + \mathcal{P}_{\mathcal{K}}(\bar{D}A) + \mathcal{P}_{\mathcal{K}}(\bar{D}\bar{A}) = 1 \tag{8}$$

$$\mathcal{P}_{\mathcal{K}}(DA) \geq 0, \mathcal{P}_{\mathcal{K}}(D\bar{A}) \geq 0, \mathcal{P}_{\mathcal{K}}(\bar{D}A) \geq 0, \mathcal{P}_{\mathcal{K}}(\bar{D}\bar{A}) \geq 0 \tag{9}$$

So $\mathcal{IM}_{umi}(\mathcal{K}_1) = 0.4$, similarly $\mathcal{IM}_{umi}(\mathcal{K}_2) = 0.45$

V. Conclusion

In this paper, we have investigated the most common inconsistency measures for logical, probabilistic-logical framework and adapted them to the probabilistic framework. In proportion to each inconsistency measure, we provided the assessment of desirable properties and then illustrated by several examples. For future work we also plan to apply the work reported here for buiding a set of belief merging operators. Then we will go on investigating to apply these operators to merge a set of (jointly) inconsistent probabilistic knowledge bases into a consistent one.

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References

[1] N. T. Nguyen, *Advanced Methods for Inconsistent Knowledge Management:* Springer-Verlag New York, Inc, 2008.
[2] V.D. Nguyen and M.G. Merayo, *Intelligent collective: some issues with collective cardinality,* . Journal of Information and Telecommunication, DOI: http://dx.doi.org/10.1080/24751839.2017.1323702, 2017.
[3] M. Finthammer, G. Kern-Isberner and M. Ritterskamp, *Resolving inconsistencies in probabilistic knowledge bases,* In KI 2007, pages 114-128 , 2007.
[4] G. Kern-Isberner, *Conditionals in Nonmonotonic Reasoning and Belief Revision,* Number 2087 in Lecture Notes in Computer Science, Springer, 2001.
[5] M. Thimm, *Measuring inconsistency in probabilistic knowledge bases,* In Bilmes, J., and Ng, A. Y., eds., Proc. Uncertainty in Artificial Intelligence (UAI 2009), pages 530-537, AUAI Press, 2009.
[6] D. Picado-Muino, *Measuring and repairing inconsistency in probabilistic knowledge bases,* International Journal of Approximate Reasoning 52(6), pages828-840, 2011.
[7] M. Thimm, *Inconsistency measures for probabilistic logics,* Artificial Intelligence 197, pages 1-24, 2013.
[8] J. Grant and A. Hunter, *Distance-based measures of inconsistency,* In Proceedings EC SQARU2013, pages 230-241, 2013.
[9] G. D. Bona, *Measuring inconsistency in probabilistic knowledge bases,* PhD thesis, University of Sao Paulo, 2016.
[10] A. Hunter and S. Konieczny, *Shapley inconsistency values,* In: Proceedings of the 10th International Conference on Knowledge Representation (KR06), pp. 249-259. AAAI Press, 2006.
[11] S. Konieczny, and S. Roussel, *A reasoning platform based on the MI shapley inconsistency value,* In van der Gaag, L. C., ed., Proc. Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU 2013, volume 7958 of Lecture Notes in Computer Science, pages 315-327. Springer, 2013.
[12] J. Grant and A. Hunter, *Measuring the good and the bad in inconsistent information,* In Proceedings IJCAI 2013, 2013.
[13] A. Hunter and S. Konieczny, *Measuring inconsistency through minimal inconsistent sets,* In Brewka, G., and Lang, J., eds., Proc. Principles of Knowledge Representation and Reasoning, KR 2008, 358-366. AAAI Press, 2008.
[14] J.Grant, A. Hunter, *Measuring consistency gain and information loss in stepwise inconsistency resolution,* In: Proceedings of the 11th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2011), no. 6717 in LNAI, pages 362-373, Springer, 2011.
[15] N. Potyka, *Linear Programs for Measuring Inconsistency in Probabilistic Logics,* Proceeding KR 2014 (KR14), pages 568-578. AAAI Press, 2014.
[16] N. Potyka and and M. Thimm, *Probabilistic reasoning with inconsistent beliefs using inconsistency measures,* International Joint Conference on Artificial Intelligence 2015 (IJCAI15), pages 31563163, 2015.
[17] N. Potyka and and M. Thimm, *Consolidation of probabilistic knowledge bases by inconsistency minimization,* In Proceedings ECAI 2014, pages 729-734, IOS Press, 2014.
[18] D. Nute and C. Cross, *Conditional logic,* In D. Gabbay and F. Guenther, editors, Handbook of Philosophical Logic, volume 4, pages 1-98. Kluwer, 2002.
[19] W. Rodder and L. Xu, *Elimination of inconsistent knowledge in the probabilistic expertsystem-shell spirit (in german),* In Proc. Symposium Operations Research 2000, pages 260-265, 2001.
[20] L. S. Shapley, *A value for n-person games,* In H.W.Kuhn and A.W. Tucker, editors, Contributions to the Theory of Games II, volume 28 of Annals of Mathematics Studies, pages 307-317, Princeton University Press, 1953.
[21] V. D. Nguyen and N. T. Nguyen, *An influence analysis of the inconsistency degree on the quality of collective knowledge for objective case,* Proc ACIIDS, 2, pages 23-32, 2016.