A New Windowed Graph Fourier Transform

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***Abstract*—Many practical networks can be mathematically modeled as graphs. Graph signal processing (GSP), intersecting graph theory and computational harmonic analysis, can be used to analyze graph signals. Just as short-time Fourier transform (STFT) for time-frequency analysis in classical signal processing, we have windowed graph Fourier transform (WGFT) for vertex-frequency analysis in GSP. In this paper, we introduced a new graph modulation operator that satisfies the property of spectral conservation, and a new graph translation operator with interesting properties. Based on these operators, we presented a new method to obtain the WGFT with a tight vertex-frequency frame. These GSP tools were developed based on the graph adjacency matrix. Using time-series graph, USA graph and random graph as examples, we showed by simulation the advantages of our proposed GSP tools over the state-of-the-arts.**

***Index Terms*—Graph signal processing, vertex-frequency analysis, vertex-frequency Fourier transform, graph modulation, graph translation.**

I. INTRODUCTION

Practical networks, such as public transport, mobile-user, social, sensor, and biological networks, can be mathematically modeled as graphs. Hence, they provide us with data on graphs, where data are represented as to reside on graph vertices, and have been studied under graph theory [1]. A new way to look at graph data is to see them as "signals". As a consequence, *graph signal processing* (GSP) has emerged [2], [3], which can be seen as intersection of graph theory and computational harmonic analysis. It is of interest to generalize typical signal processing tasks (representation, frequency analysis, filtering, detection, estimation, separation, etc.) for graph signals.

Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ denotes the set of vertices of the graph with $|\mathcal{V}| = N$, and $\mathcal{E}$ the set of edges. A signal defined on $\mathcal{G}$ is a vector $\mathbf{f} \in \mathbb{R}^N$ whose n^{th} component represents the signal value at the n^{th} vertex of $\mathcal{G}$. Associated with $\mathcal{G}$ are two special matrices called the Laplacian matrix ($\mathbf{L}$) and the adjacency matrix ($\mathbf{A}$).

Just as Fourier transform (FT) and short-time Fourier transform (STFT) for frequency and time-frequency analyses in classical signal processing (CSP) [4], we also have graph Fourier transform (GFT) and windowed graph Fourier transform (WGFT) for frequency and vertex-frequency analyses in GSP. In this paper, we are interested in the WGFT that provides a localized spectral analysis of graph signals.

Recall that by focusing on a signal $f(t)$ at a region around time a and frequency k using a time-frequency localized window $g_{a,k}(t)$, the STFT of $f(t)$ is defined as

$$S_f(a, k) \triangleq \langle f, g_{a,k} \rangle. \tag{1}$$

The localized window is obtained by translating the original window $g(t)$ in time by a and then modulating the result by e^{jkt}; that is, $g_{a,k}(t) = g(t - a)e^{jkt}$. Therefore, it is natural to first define in GSP the graph (or generalized) translation and modulation operators, in order to define the WGFT.

By aiming to design a window that is localized in the vertex-frequency domain, there have been so far two methods proposed for the WGFT. First, Shuman, Ricaud, and Vandergheynst defined the WGFT via introducing the graph convolution, translation and modulation operators [5]. Inspired by the equivalence of convolution in time is equivalent to multiplication in frequency in CSP, their graph translation operator, translating the window by a vertices in the vertex domain, was defined as the inverse GFT of the multiplication of the GFT of the window with the GFT of the delta function centred at vertex a. The Laplacian matrix of the underlying graph is used in the definition. Here, the window is not defined first but generated from a frequency kernel; hence the operator is called "kernelized" operator. An interesting property of this translation operator is that the magnitude of the translated window decays with vertex distance away from the shifted location, thus helping design well-localized windows. However, the operator is neither shift-invariant nor isometric in general, causing a significant change in the frequency content of the signal after translation. Inspired by the fact that modulation is equivalent to multiplication by a Laplacian eigenfunction in CSP, their graph modulation operator was defined as multiplication of the window with an eigenvector of the Laplacian matrix. However, the property in CSP that modulation in time corresponds to translation in frequency does not generally hold in GSP. In addition, their graph modulation causes changes to the energy content of the signals. Moreover, the frame related to their WGFT is not tight.

Second, Tepper and Sapiro in [6] takes a different approach for the graph localization and modulation operators in defining the WGFT, inspired from the existing notion of the personalized PageRank (PPR) vector [7] in local spectral graph theory, useful in the problem

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of community detection.In particular, the PPR vector, defined based on the normalized Laplacian matrix, provides a natural way to find a community localized around a set of seed vertices by finding other vertices that have stronger relationship to the seed vertices than the rest of the graph. The PPR vector associated to a given seed set of vertices behaves as a localization of the graph signal around the seed vertices. By seeing this interesting connection between the local spectral graph theory and our underlying problem of localized spectral analysis of graph signals, they directly defined the window localized at vertex a by the PPR vector that is constructed based on a seed set that contains only the vertex a. This definition of a localization operator is obtained directly rather than defining first an original window and then a graph translation operator to translate it. The localization operator can be recognized as a smooth signal-dependent window kernel as in CSP. The "signal-dependent" part is due to the inclusion of the graph degree matrix in the normalized Laplacian matrix. The "smooth" part is provided thanks to the way the PPR optimally calculates the signal weights/values at the vertices of the local community other than the seed vertices. However, since their localization operator depends on the signal, it is generally not isometric. In addition, it is quite computationally demanding because it has to solve an optimization problem to obtain the PPR vector. Their graph modulation operator is defined similar to that by Shuman et al., but with the use of the normalized Laplacian matrix instead. It is not clear if the frame stability condition is satisfied in order to recover the graph signal from its spectrogram.

In general, both methods only focused on designing the vertex-frequency localized window for the WGFT rather the graph translation and modulation operators for *arbitrary* graph signals. In addition, neither of them can be applied to signals defined over directed graphs because the Laplacian matrix is only defined for undirected graphs. The drawbacks of the graph translation operators defined in [5], [6] suggested the formulation of alternative definitions. Sandryhaila *et al.* defined it based on the adjacency matrix [8]. Girault *et al.* defined an isometric graph translation operator [9]. Unfortunately, both face some drawbacks when we compare to its counterpart in CSP including the isometric property, energy conservation or computational complexity.

The drawbacks and circumstances related to graph translation and modulation operators as explained above encourage us to propose in this paper new ways to define them and hence the WGFT. We are inspired by a recent definition by Gavili and Zhang in [10] of an optimal graph shift operator that takes the advantages of [8], [9]. The contribution of the paper is three-fold. First, we introduce a new optimal graph modulation operator that satisfies the property of spectral conservation. Second, based on [10], we define a graph translation operator with nice properties. Third, we present a new method to obtain the windowed graph Fourier dictionary with a tight vertex-frequency frame.

In addition, unlike the use of the Laplacian matrix by [2], [11], we use the adjacency matrix on which the GFT was defined [8], [12]. The exploitation of the adjacency matrix is due to the fact that this matrix is highly useful to present various types of graphs, including undirected, directed, connected and weighted graphs.

II. Proposed Methods

A. New Graph Modulation Operator

A time series signal $f(t)$ is to be modulated by a modulation operator M_ε to yield the modulated signal

$$M_\varepsilon f(t) \triangleq f(t)e^{j\varepsilon t}, \tag{2}$$

indicating that $e^{j\varepsilon t}$ is an eigenfunction under M_ε. By taking the Fourier transform on both sides of (2), the modulation operator can also be seen as translating the signal in the spectral domain

$$\widehat{M_\varepsilon f}(\omega) = \hat{f}(\omega - \varepsilon), \tag{3}$$

where $\widehat{x}$ denotes the spectrum of x.

We now wish to define a generalized modulation operator in GSP.

Denote by χ_l and λ_l, for $l = 0, \ldots, N-1$, the eigenvectors and eigenvalues of $\mathbf{A}$. The *distinct* eigenvalues are called the graph frequencies and they form the graph frequency spectrum: $\sigma(\mathbf{A}) = \{\lambda_0, \lambda_1, \ldots, \lambda_{M-1}\}$, where $0 < M \leq N$. In this paper, due to the restriction of paper length, we assume that all the eigenvalues are distinct, i.e., $M = N$. Formulation with respect to the case when $M < N$ can be done, based on the block formulation of the GFT in [12]. Accordingly, given the eigenvalue decomposition $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, then $\mathbf{V}^{-1}$ is called the GFT matrix. Hence, for a graph signal $\mathbf{f}$, its GFT at graph frequency λ_a is given by

$$\hat{\mathbf{f}}(\lambda_a) \triangleq \sum_{n=1}^{N} \mathbf{V}^{-1}(a, n)\mathbf{f}(n), \tag{4}$$

or over the whole spectrum by

$$\hat{\mathbf{f}} \triangleq \mathbf{V}^{-1}\mathbf{f}. \tag{5}$$

Denote by $\mathbf{M}_k$ the graph modulation operator at frequency k and apply it to the graph signal $\mathbf{f}$ to obtain the modulated signal in the vertex domain

$$\tilde{\mathbf{f}}_k = \mathbf{M}_k\mathbf{f}. \tag{6}$$

Similar to (3), the modulated signal in the frequency domain is then expressed as

$$\widehat{\mathbf{M}_k\mathbf{f}}(\lambda_a) = \widehat{\mathbf{f}}(\lambda_{(a-k)_N}). \tag{7}$$

where $(a-k)_N \triangleq (a-k) \mod N$. In this way, the frequency content of $\mathbf{f}$ at λ_i has been translated to a new location at $\lambda_{(i+k)_N}$. Using (4), we obtain

$$\widehat{\tilde{\mathbf{f}}}_k(\lambda_a) = \widehat{\mathbf{f}}(\lambda_{(a-k)_N}) = \sum_{n=1}^{N} \mathbf{V}^{-1}((a-k)_N, n)\mathbf{f}(n),$$

or more compactly

$$\hat{\tilde{\mathbf{f}}}_k = \mathbf{P}^k\mathbf{V}^{-1}\mathbf{f}, \tag{8}$$

where $\mathbf{P}$ the $N \times N$ cyclic permutation matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Here, $\mathbf{P}^k$ cyclically permutes the GFT matrix, $\mathbf{V}^{-1}$, upward k rows.

Going back to the vertex domain, this modulated signal is then expressed by its inverse GFT as

$$\begin{aligned} \tilde{\mathbf{f}}_k &\triangleq \mathbf{V}\hat{\tilde{\mathbf{f}}}_k \\ &= \mathbf{V}\mathbf{P}^k\mathbf{V}^{-1}\mathbf{f} \\ &= \mathbf{V}\mathbf{V_P}\mathbf{\Lambda}_\mathbf{P}^k\mathbf{V_P}^{-1}\mathbf{V}^{-1}\mathbf{f}, \\ &= \mathbf{V_M}\mathbf{\Lambda}_\mathbf{M}^k\mathbf{V}_\mathbf{M}^{-1}\mathbf{f}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\mathbf{P} = \mathbf{V_P}\mathbf{\Lambda_P}\mathbf{V_P}^{-1} \quad \text{(Eigenvalue decomposition)}, \\ &\mathbf{V_M} := \mathbf{V}\mathbf{V_P}, \\ &\mathbf{\Lambda_M} := \mathbf{\Lambda_P} = \operatorname{diag}\left(1, e^{-j2\pi\frac{1}{N}}, ..., e^{-j2\pi\frac{N-1}{N}}\right). \end{aligned}$$

Denote

$$\mathbf{M} := \mathbf{V_M}\mathbf{\Lambda_M}\mathbf{V}_\mathbf{M}^{-1} \tag{10}$$

Then, by comparing (6) with (9), we can define the graph modulation operator as

$$\mathbf{M}_k \triangleq \mathbf{M}^k = \mathbf{V_M}\mathbf{\Lambda}_\mathbf{M}^k\mathbf{V}_\mathbf{M}^{-1}, \tag{11}$$

which is an optimal solution.

Interestingly, in the case of graph time-series, $\mathbf{M}$ is exactly the circular frequency-shift operator in CSP, that is,

$$\left(\mathbf{M}_k\mathbf{f}\right)(n) = e^{\frac{j2\pi n}{N}k}\mathbf{f}(n). \tag{12}$$

Proof. From (12), the modulation matrix $\mathbf{M}_k$ can be formed by

$$\mathbf{M}_{\text{time-series}} = \operatorname{diag}\left(1, e^{j2\pi\frac{1}{N}}, ..., e^{j2\pi\frac{N-1}{N}}\right).$$

In addition, since $\mathbf{P}$ is also diagonalized by the discrete Fourier transform (DFT) matrix, $\mathbf{F}_N^{-1}$, we have

$$\mathbf{P} = \mathbf{F}_N \operatorname{diag}\left(1, e^{j2\pi\frac{1}{N}}, ..., e^{j2\pi\frac{N-1}{N}}\right)\mathbf{F}_N^{-1}.$$

Therefore,

$$\begin{aligned} \tilde{\mathbf{f}}_k &= \mathbf{V}\mathbf{P}^k\mathbf{V}^{-1}\mathbf{f} \\ &= \mathbf{F}_N^{-1}\mathbf{F}_N\,\mathbf{M}^k_{\text{times-series}}\,\mathbf{F}_N^{-1}\mathbf{F}_N\mathbf{f} \\ &= \mathbf{M}^k_{\text{time-series}}\mathbf{f}. \end{aligned}$$

□

B. New Graph Translation Operator

On a time series signal $f(t)$, the translation operator in CSP, defined for a translation amount of a in time, denoted by T_a, is done via the change of variable: $(T_af)(t) \triangleq f(t-a)$. On a graph signal $\mathbf{f}$, a number of methods have been proposed to define a graph translation operator, $\mathbf{T}_a$. For example, as explained in Section I, Shuman *et al.* introduced a graph translation operator as a "kernelized" operator, based on the observation in CSP that translation can be seen as convolution of the underlying signal with a delta function centred at time translated time: $T_af(t) \triangleq (f * \delta_a)(t)$.

In defining our graph translation operator in this paper, we are inspired by results derived from [10]. In particular, the shift operator is considered as a linear map $\mathbf{A}_\phi$ obtained from the adjacency matrix

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}_h\mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda}_\phi\mathbf{V}^{-1} = \mathbf{A}_h\mathbf{A}_\phi,$$

where $\mathbf{\Lambda}_\phi = \mathbf{\Lambda_M}$. The shifted signal is given by $\tilde{\mathbf{f}}_1(n) = \mathbf{A}_\phi\mathbf{x}(n)$. Thus, a-step translated version of the original signal $\mathbf{x}$ can be given by

$$\mathbf{f}_a(n) = \mathbf{A}_\phi^a\mathbf{f}(n) = \mathbf{V}\mathbf{\Lambda}_\phi^a\mathbf{V}^{-1}\mathbf{f}(n). \tag{13}$$

Therefore, we can define the graph translation operator as

$$\mathbf{T}_a \triangleq \mathbf{A}_\phi^a = \mathbf{V}\mathbf{\Lambda}_\phi^a\mathbf{V}^{-1}. \tag{14}$$

In this way, the graph translation operator has several important properties. In particular, for all $\mathbf{f} \in \mathbf{L}^2$, we obtain a list of properties of the generalized translation operator shown as in Table I. Proofs of the properties are omitted due to the limitation of the paper length.

TABLE I: Properties of Graph Translation

No	Property	Description
1	Isometry	$d(\mathbf{T}_a\mathbf{x}, \mathbf{T}_a\mathbf{y}) = d(\mathbf{x}, \mathbf{y})$
2	Energy Conservation	$\lVert\mathbf{T}_a\mathbf{f}\rVert^2 = \lVert\mathbf{f}\rVert^2$
3	Invariance	$\mathbf{T}_a(\mathbf{f} * \mathbf{g}) = (\mathbf{T}_a\mathbf{f}) * \mathbf{g} = \mathbf{f} * (\mathbf{T}_a\mathbf{g})$
4	Commutativity	$\mathbf{T}_a(\mathbf{T}_b\mathbf{f}) = \mathbf{T}_b(\mathbf{T}_a\mathbf{f}) = \mathbf{T}_{a+b}\mathbf{f}$
5	Circular	$\mathbf{T}_N\mathbf{f} = \mathbf{A}_\phi^N\mathbf{f} = \mathbf{f}$

In addition, we also get some other nice properties:
6) Fourier's basis of $\mathbf{A}_\phi$ and $\mathbf{A}$ are the same, thus frequency contents of graph signals are safe.
7) In the case for graph time-series signals (ring graphs), the graph translation operator is analogous to its counterpart in CSP.

C. New Windowed Graph Fourier Transform

With the above newly defined graph modulation and translation operators, we introduce in this section a new definition of the WGFT, using a Fourier dictionary $\mathbf{D} = \{\mathbf{g}_{a,k}\}$, where

$$\mathbf{g}_{a,k} := \mathbf{M}_k\mathbf{T}_a\mathbf{g} = \mathbf{M}^k\mathbf{A}_\phi^a\mathbf{g}, \tag{15}$$

with $\mathbf{g}$ is a window. The resulting projection of a graph signal $\mathbf{f}$ on each subdictionary atom $\mathbf{g}_{a,k}$ is then given by

$$\mathbf{S}_\mathbf{f}(a,k) = \langle\mathbf{f}, \mathbf{g}_{a,k}\rangle. \tag{16}$$

Now, we recall the necessary and sufficient frame condition to recover a signal (vector) $\mathbf{f}$ from its inner product with a dictionary $\Phi \in \mathbb{R}^{N,M}$ [13], that is: its frame has to give an energy equivalence if there exist constants $\beta \geq \alpha > 0$ such that

$$\alpha\|\mathbf{f}\|^2 \leq \sum_n |\langle \mathbf{f}, \phi_n\rangle|^2 \leq \beta\|\mathbf{f}\|^2, \quad \forall \mathbf{f} \in \mathbb{R}^N. \tag{17}$$

Denote $\mathbf{A}_{\text{tr}}^{(a,k)} := \mathbf{M}^k \mathbf{A}_\phi^a$, then

$$\begin{aligned}\mathbf{A}_{\text{tr}} &= \mathbf{V}_{\mathbf{M}}\mathbf{\Lambda}_{\mathbf{M}}^k\mathbf{V}_{\mathbf{M}}^{-1}\mathbf{V}\mathbf{\Lambda}_\phi^a\mathbf{V}^{-1}\\ &= \mathbf{V}\mathbf{P}^k\mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda}_\phi^a\mathbf{V}^{-1}\\ &= \mathbf{V}\mathbf{P}^k\mathbf{\Lambda}_\phi^a\mathbf{V}^{-1}\\ &= \mathbf{V}\mathbf{\Lambda}_{\phi(k)}^a\mathbf{V}^{-1}.\end{aligned}$$

Hence, the magnitude of eigenvalues of $\mathbf{A}_{\text{tr}}^{(a,k)}$ is always equal 1 for all a and k, thanks to the above nice properties of modulation (k) and translation (a): $\lambda_i = e^{j\phi_i'}$.

As previously expressed, the vector direction does not change when the transform is applied. That means $\mathbf{A}_{\text{tr}}^{(a,k)}$ is an isometric linear map, so $\|\mathbf{A}_{\text{tr}}^{(a,k)}\mathbf{x}\|^2 = \|\mathbf{x}\|^2$, for any $\mathbf{x}$. Therefore,

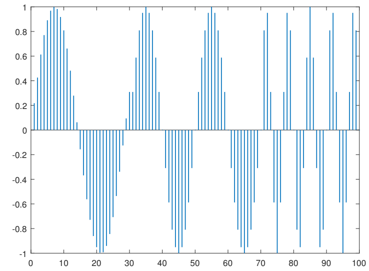
$$\begin{aligned}\mathbf{S} &= \sum_{a,k=1}^{N} |\langle \mathbf{f}, \mathbf{g}_{a,k}\rangle|^2\\ &= \sum_{a,k=1}^{N} |\langle \mathbf{f}, \mathbf{M}_k\mathbf{T}_a\mathbf{g}\rangle|^2\\ &= \sum_{a,k=1}^{N} |\langle \mathbf{f}, \mathbf{A}_{\text{tr}}^{(a,k)}\mathbf{g}\rangle|^2\\ &= \sum_{n=1}^{N} |\mathbf{f}(n)|^2 \sum_{a,k=1}^{N} (\mathbf{A}_{\text{tr}}^{(a,k)}\mathbf{g})(n)^2\\ &= \sum_{n=1}^{N} |\mathbf{f}(n)|^2 \sum_{a,k=1}^{N} (\mathbf{A}_{\text{tr}}^{(a,n)}\mathbf{g})(k)^2\\ &= \sum_{n=1}^{N} |\mathbf{f}(n)|^2 \sum_{a=1}^{N} \|\mathbf{A}_{\text{tr}}^{(a,n)}\mathbf{g}\|^2\\ &= N\|\mathbf{g}\|^2\|\mathbf{f}\|^2.\end{aligned}$$

As a result, $\alpha = \beta = N\|\mathbf{g}\|^2$ that leads to the conclusion that the Fourier frame is tight. In other words, the inverse WGFT can be implemented by the synthesis operator using the dot (inner) product of the spectrogram of the graph signal and its window. The same way to express the above analysis operator, we also get the synthesis

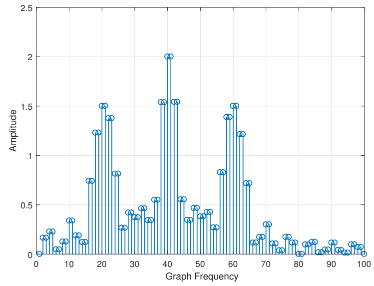
$$\sum_{a,k=1}^{N} \langle \mathbf{f}, \mathbf{g}_{a,k}\rangle \mathbf{g}_{a,k}(n) = N\|\mathbf{g}\|^2\mathbf{f}(n),$$

and hence

$$\mathbf{f}(n) = \frac{1}{N\|\mathbf{g}\|^2}\sum_{a,k=1}^{N} \mathbf{S}_{\mathbf{f}}(a,k)\mathbf{g}_{a,k}(n). \tag{18}$$



(a) Graph signal, $\mathbf{f}$



(b) Its spectrum, $\hat{\mathbf{f}}$

Fig. 1: Time-series graph and its spectrum.

To sum up, in GSP, the windowed Graph Fourier dictionary $\mathbf{D} = \{\mathbf{g}_{a,k}\}$ is generalized with stable analysis and synthesis operators.

III. Simulations

In this section, we provide simulated experiments to illustrate the ability of our proposed methods. Three examples of graph signals are considered: (i) a time-series graph (a.k.a. path graph), (ii) a USA graph, and (iii) a random graph.

In particular, the time-series graph is composed of three components localized in three different locations, constructed by $\mathbf{f} = [\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3]$ in which $\mathbf{f}_1 = \sin(\omega_0 n), \mathbf{f}_2 = \sin(2\omega_0 n)$, and $\mathbf{f}_3 = \sin(3\omega_0 n)$ with $\omega = 2\pi/7$, as shown in Figure 1. The USA graph contains the temperature measurements for one year in 150 cities across the USA, which has been detrended for our purpose, as shown in Figure 2. The random graph is randomly generated with 100 nodes, as shown in Figure 3.

A. Results for Graph Modulation Operator

It can be seen in the Figures 4 and 5 that our graph modulation is much the same as its counterpart in CSP. In particular, our proposed method yields new signals with their spectrum that are kept unchanged, as compared to their original form. Meanwhile, Shuman's method [14] do not hold the core information, the modulated signals are disintegrated into lots of frequency components. The method may be suitable for monocomponent signals only; we refer the reader to [14] for further details.

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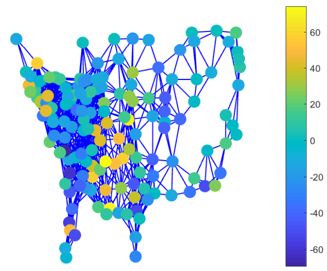
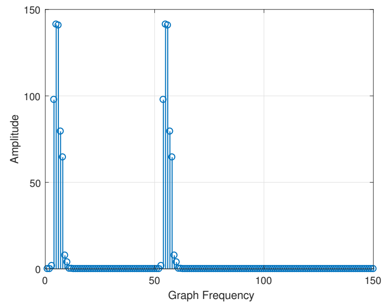
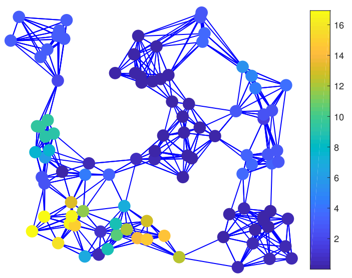


(a) Graph signal, $\mathbf{f}$

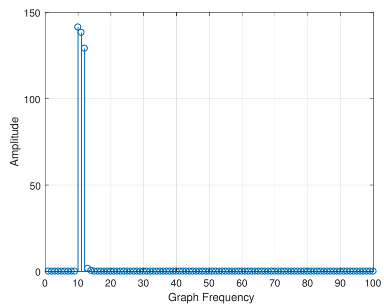


(b) Its spectrum, $\hat{\mathbf{f}}$

Fig. 2: USA graph and its spectrum.

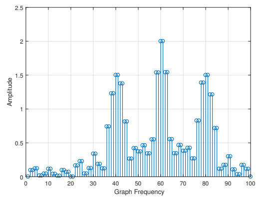


(a) Graph signal, $\mathbf{f}$

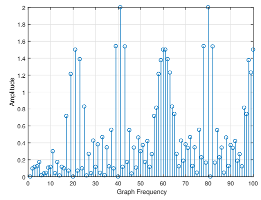


(b) Its spectrum, $\hat{\mathbf{f}}$

Fig. 3: Random graph and its spectrum.

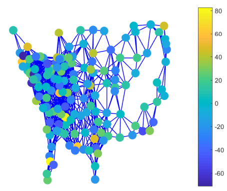


(a) Our method: $\widehat{\mathbf{M}_{20}\mathbf{f}}$

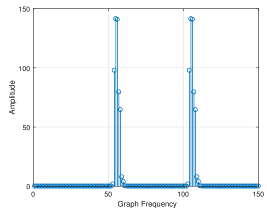


(b) Shuman's method: $\widehat{\mathbf{M}_{20}\mathbf{f}}$

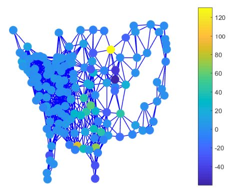
Fig. 4: Modulation of the time-series graph; $k = 20$.



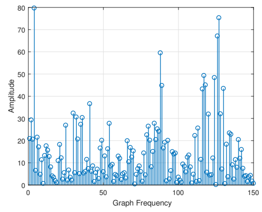
(a) Our method: $\mathbf{M}_{50}\mathbf{f}$



(b) Our method: $\widehat{\mathbf{M}_{50}\mathbf{f}}$

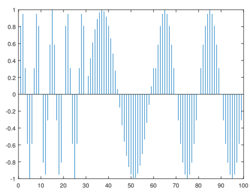


(c) Shuman's method: $\mathbf{M}_{50}\mathbf{f}$

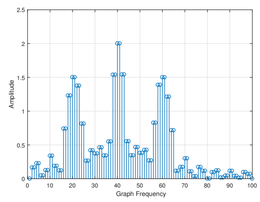


(d) Shuman's method: $\widehat{\mathbf{M}_{50}\mathbf{f}}$

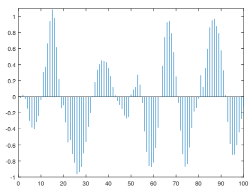
Fig. 5: Modulation of the USA graph; $k = 50$.



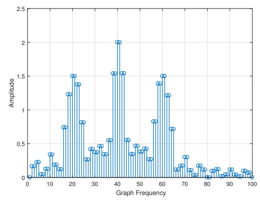
(a) Our Method: $\mathbf{T}_{30}\mathbf{f}$



(b) Spectrum $\widehat{\mathbf{T}_{30}\mathbf{f}}$



(c) Shuman's Method: $\mathbf{T}_{30}\mathbf{f}$

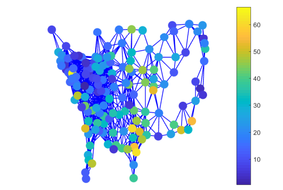


(d) Spectrum $\widehat{\mathbf{T}_{30}\mathbf{f}}$

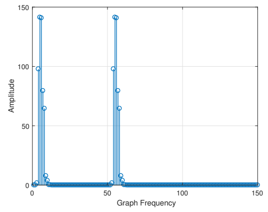
Fig. 6: Translation of the time-series graph; $a = 30$.

B. Results for Graph Translation Operator

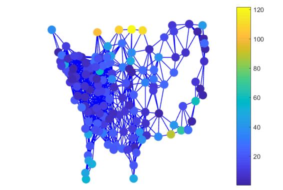
The results in Figures 6 and 7 illustrate that there are no difference in spectral representation of the original signals and the translated ones using our method. In the case of the time-series graph, our graph translation shares the same properties with those of the shift operator in CSP. The experiment with the USA graph also demonstrates the effectiveness of our method in which all properties of the graph translation operator are held, such as energy, frequency content conservation, invariance and so forth; this is not true in general with Shuman's method.



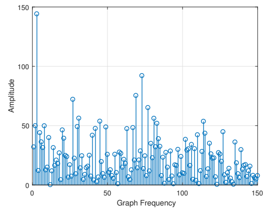
(a) Our Method: Signal $\mathbf{T}_{30}\mathbf{f}$



(b) Spectrum $\widehat{\mathbf{T}_{30}\mathbf{f}}$

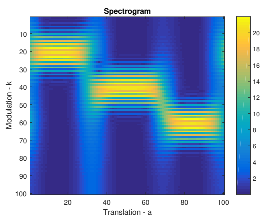


(c) Shuman's Method: $\mathbf{T}_{30}\mathbf{f}$

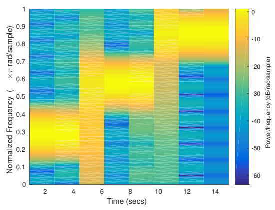


(d) Spectrum $\widehat{\mathbf{T}_{30}f}$

Fig. 7: Translation of the USA graph; $a = 30$.

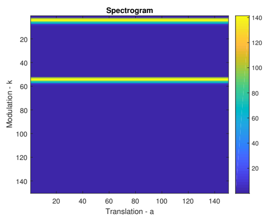


(a) WGFT

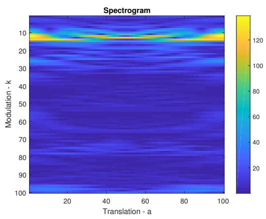


(b) STFT

Fig. 8: WGFT and STFT of the time-series graph.



(a) USA graph



(b) Random graph

Fig. 9: WGFT of the USA graph and the random graph.

C. *Results for Windowed Graph Fourier Transform*

Consider a graph time-series signal having 100 nodes and a window $\hat{g}(\lambda) = Ce^{-\alpha\lambda}$, where α and C are constant. In the case $\alpha = 300$ and $C = 1$, the resulting spectrogram in Figure 8 shows that our new WGFT generates three difference bands restricted to the three segments of the signal, which is analogous to the STFT in CSP. Experiments with the USA graph and the random graph also illustrates the success of our method, as shown in Figure 9.

IV. Conclusion

In this study, thanks to the properties of cyclic permutation matrix $\mathbf{P}$, we have introduced a new way to define two fundamental concepts in GSP: graph translation and graph modulation. Based on that, we have proposed a new definition for the WGFT such that it successfully possess similar behavior to the STFT– its counterpart in CSP. We have showed that our methods overcome the drawbacks of the existing methods and can also be applied to specific applications in particular and to analyzing any graph signals in general.

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