Noname manuscript No.
(will be inserted by the editor)

Fractional Frequency Reuse in Multi-Tier Networks: Performance Analysis and Optimization

Sinh Cong Lam · Kumbesan Sandrasegaran

Received: date / Accepted: date

Abstract Frequency Reuse (FR) is an efficient approach to improve the network performance in a multi-cell cellular network. In this paper, we investigate the performance of heterogeneous networks utilising two well-known frequency reuse algorithms, called Strict FR and Soft FR, with a reuse factor of Δ ($\Delta > 1$). Based on a two-phase operation of the FR algorithm, we develop a new approach to analyse cellular networks based on the Poisson Point Process (PPP) model which successfully demonstrates the impact of the network parameters (such as the number of allocated Resource Blocks (RBs) and bias factor) and FR parameters (such as Base Station (BS) transmit power) on the performance of a user and overall network. Compared to related works, we propose the following novel approaches: (i) we investigate flexible FR networks in which users have connections with the BSs which deliver the highest performance; (ii) the BS observes SINR on the data channel to classify each user into either a Cell-Edge User (CEU) or Cell-Center User (CCU); (iii) the initial state of the network is considered to establish the initial network interference when new users arrive and request connections to the BSs. In the case of a single-tier network, it is proved that our analytical approach is more accurate than previous works. In the case of a two-tier network, the analytical results indicate that compared to the 3GPP model, our proposed model not only reduces up to 40.79% and 3.8% power consumption of a BS on the data channel but also achieves 16.08% and 18.63% higher data rates in the case of Strict FR and Soft FR respectively. Furthermore, the paper presents an approach to find an optimal value of SINR thresholds and bias factor to achieve the maximum network performance.

Keywords PPP cellular network, heterogeneous cellular network, coverage probability, frequency reuse.

Sinh Cong Lam
VNU - University of Engineering and Technology
Faculty of Electronics and Telecommunication

Kumbesan Sandrasegaran
University of Technology, Sydney
Faculty of Engineering and Information Technology

1 Introduction

In LTE networks, in order to improve spectrum efficiency, adjacent cells are allowed to use the same frequency band, and this is referred to reuse of frequency. However, reuse of frequencies with a high density leads to an increase of Inter-Cell Interference (ICI), especially in heterogeneous networks with macro cells and pico cells, which impacts the performance of mobile users, especially for Cell-Edge Users (CEUs) who experience low Signal-to-Interference-plus-Noise Ratios (SINRs). InterCell Interference Coordination (ICIC) techniques such as FR algorithm have been introduced to control the reuse of frequencies and related network factors in order to mitigate ICI [1,2]. Generally, the operation of Frequency Reuse (FR) scheme can be separated into two phases. During the first phase, called *establishment phase*, the user measures the SINR and reports to the serving BS. Based on the reported SINR, the BS classifies each associated user into either CCU or CEU. After that, connection between the user and the BS is established and data is transferred during the second phase, called *communication phase*. This paper focuses on studying the performance of FR in downlink multi-tier cellular networks with multi-Resource Blocks (RBs) and multi-users.

In our recent work [3], we conducted a study on performance of CCU and CEU in the uplink cellular single-tier networks using FR. In this work, the interfering BSs of a user are classified into two independent groups in which the first group consits of BSs transmitting on the CC power and the second group consists of BSs transmitting on the CE power. In addition, a two-phase operation of FR was discussed for both CCU and CEU according to 3GPP recommendations in this work, in which the BS uses the SINR on the uplink control channel to classify each user into either CCU or CEU during the establishment phase, which is followed by data transmission process between the user and it's serving BS during the communication phase. Thus, the average coverage probabilities of the CCU and CEU were defined as the conditional probability of the SINR during the communication phase given SINR during the establishment phase. Motivated by these works, we have developed a model based on PPP to analyse performance of the multi-tier networks using FR.

The work in this paper can be distinguished from our previous work in [3] and related well-known results in [4] in four major aspects. *First*, we consider the FR cellular network with flexible cell association in which the user connects to a BS with the highest average received signal power. In our previous work and related work in [4], the authors assumed that the user associates with the nearest BS, which is not appropriate for multi-tier networks. Since the channel power gain and path loss exponents in multi-tier networks may vary between cells and tiers, the nearest BS may be strongly affected by fading and path loss, and consequently it may not deliver high user performance as other BSs. *Second*, we assume that the BS uses SINR on the data channel to classify users into CCUs and CEUs. Every BS is continuously transmitting downlink control information, and subsequently each control channel experiences the ICI from all adjacent BSs. Thus, the measured SINR on the control channel does not depend on the network status such as the average number of RBs and number of users. In contrast, the BS transmits on the data channel only if the user requires data from the BS. Hence, the measured SINR on the data channel strongly depends on the network status and consequently, this can provide more accuracy on the ICI in the network. *Third*, we define the initial state of the networks, when the BSs are activated to serve a particular number of existing users and create an amount of initial network ICI. All new users come and request connections with the networks are affected by this ICI. This assumption is suitable for real networks since there are always existing users and ICI in the networks. *Forth*, compared to [5,6], we develop the analytical approach, which separately but correlatively evaluates interference from BSs transmitting on CC and CE RBs in our previous work [3], for the downlink. The main contributions of the paper are summarized as follows:

– We derive the highly tractable expressions of average cell data rate, number of CCUs, number of CEUs, and CCU and CEU performance in terms of the average coverage probability and average data rate.
– The effects of SINR threshold, bias factor on the network performance are analysed with respects to number of user and number of RBs.
– By using Monte Carlo simulation, it is proved that our proposed analytical approach is more accurate than the well-known related works in [5] and [6].
– The analytical results in Section 6.2.1 indicate that our proposed model can save upto 40.79% and 3.8% power consumption of a BS on the data channel compared to 3GPP model. Furthermore, the proposed model achieves higher network data rates in both cases of Strict FR and Soft FR.
– Our analytical results indicate that with an increase in SINR threshold, the average CEU data rate increases to a peak before undergoing a rapid decline to the bottom which is followed by a slight growth. This finding contradicts the well-known results which stated that with an increase in SINR threshold, the CEU performance increased continuously in [4] and a decline followed an increase in [7]. Furthermore, our analytical results are in contrast with the conclusion about the effects of the bias factor on the network performance in [8]. Meanwhile, the author in [8] concluded that the overall network performance increased continuously with the bias factor, our results stated that the coverage network data rate increases rapidly before undergoing a decline. The differences are because the average number of users and number of RBs were not considered in previous works.

Paper Organization

The paper is organized as the following sections. Section 1 and Section 2 highlights the related research in the literture and the main contributions of this paper. The nathmatichal model of the multi-tier network, FR algorithms and scheduling algorithm are discussed in Section 3. This section also derives the average number of new CCUs and CEUs. Section 4 foucuses on computation of CCU and CEU average coverage probability. The average data rate of user and network areas are derived in Section 5. Section 6 discusses the simulation and analytical results. The interesting findings in user and network performance trends are presented in this section. Section 7 provides the conclusion of this paper which is followed by the Appendix section and references.

2 Related Works

The Poisson Point Process (PPP) network model in which the BSs are distributed as a Spatial Poisson Point Process [9] and the cell boundaries have Voronoi shapes has been selected widely to replace the conventional hexagonal model and to provide mathematical tractability. Although, the PPP network model has been studied as early as 1997 in [10], some of the important results about the performance of FR algorithms using the PPP model were presented in 2011 [5]. The user performance in terms of average coverage probability and average data rate were derived.

The authors in [11,12] extended the results of [5] for multi-tier networks and proposed methods to select optimal values of FR parameters such as SINR threshold and transmit power. Although references [5,11,12] provided the basis of analysis of FR algorithms, these works used the constant coefficient to model the ICI which is not appropriate for the PPP network model [3]. Furthermore, these works did not separate the establishment phase and communication phase for the CCU which implies that the CCU reports the measured SINR and transmits the data to the serving BS at the same time. In [3], we separated the establishment phase and communication phase by following the recommendation of 3GPP in which the BS uses the uplink SINR on the control channel for user classification purpose. In this paper, we further develop the model in [3] to obtain more accuracy of user classification in which the SINR on data channels are used for user classification purpose.

In [13,14], the FR network in which the average number of users was modelled as a Poisson RV was studied. Thus, a bias factor was introduced to handover users from a given tier to other tiers in order to maintain load balancing. In these papers, the bias factor was optimised for both uplink and downlink. Reference [15,16] presented a method to optimise the FR factor and number of BSs to maximise the system throughput as well as minimise energy consumption. In contrast to [15], the authors in [17] focused on evaluating the spectral and energy efficiencies for a given ratio of the density of BSs and users to find the optimal FR factor. Although the impact of user density was presented in [13,14,15,17,18], it was assumed that all BSs use the same transmit power which implies that there is no difference between a CCU and CEU. Moreover, these papers considered the average number of users but the impact of users on the ICI was not presented, thus it was assumed that all BSs always create ICI to users. In contrast, in order to evaluate impact of the average number of users and number of RBs on ICI, we define the allocation ratio which represents the probability that a given RB b is allocated to a user. The adjacent BS creates ICI to a user if it transmit on RB b, which is also defined as a probability. Furthermore, we define two states of the networks. During the initial state, the network is serving a given number of user and create amount of ICI during the establishment phase when the new users come and request connections with the networks. During the second state, the measured SINRs of the new users who are affected by existing ICI is considered to classify users into CCUs and CEUs.

3 System Model

3.1 Multi-Tier Network and Biased User Association

We consider a PPP cellular network with K tiers in which Tier-k ($1 \leq k \leq K$) is characterised by a density of BSs λ_k, a transmit power P_k on CC RBs and a bias factor B_k. The bias factor is used to maintain load-balancing between tiers in the heterogeneous network by handover users from a given tier to other tiers. The downlink signals in each tier experience different path loss exponents, and for Tier-k, it is denoted by α_k. The list of symbols associated with Tier-k is presented in Table 1.

Table 1 Symbol Descriptions for Tier-k

Symbol	**Meaning of Symbol**
K	Number of Tiers in a heterogeneous network.
λ_k	Density of BSs
α_k	Pathloss exponent
g_k	Channel power gain
r_k	Distance from the user to the BS
P_k	Transmit power on CC RBs
ϕ_k	Transmit power ratio between transmit power on the CE RB and CC RB
B_k	Bias factor of Tier k is an operator set parameter that is used for load-balancing in a heterogeneous network by handing over users from a heavily loaded to less loaded Tier.
A_k	Average Probability with which a user connects to a BS in Tier-k (Equation 2) [8]
T_k	SINR threshold
$\hat{T}_k$	Coverage threshold is dependent on the UE sensitivity. For both CCU and CEU, the received $SINR > \hat{T}_k$ for communication to be possible.
$M_k^{(n)}$	Average number of new users associated with a cell as defined in Equation 3

During the establishment phase, $M_k^{(o)}$ is denoted as the average number of existing users at a typical cell in Tier-k. the average number of new users who request communications with the network is modelled as a random Poisson variable with mean $\lambda^{(u)}$.

In this paper, the open access protocol is used in which a user associated with any tier is studied. However, the user may prefer a connection with a tier which has the greatest long-term average Biased-Received-Power (BRP) [8]. The BRP of the user in Tier-k is $B_k P_k r_k^{-\alpha_k} E[g_k]$ in which r_k and g_k are the distance and channel power gain between the user and the serving BS in Tier-k. In this paper, the fading channel is i.i.d Rayleigh RV with a mean of 1, i.e. $E[g_k] = 1$. Hence, the typical user is associated with a BS in Tier-k if

$$B_k P_k r_k^{-\alpha_k} = \max\left(B_j P_j r_j^{-\alpha_j}\right) \quad \text{or} \quad r_j > \left(\frac{B_j P_j}{B_k P_k}\right)^{1/\alpha_j} r_k^{\alpha_k/\alpha_j}, \quad \forall 1 \le j \le K, j \ne k \tag{1}$$

Therefore, the average probability that a user connects to a BS in Tier-k is given by [8]:

$$\begin{aligned} A_k &= \mathbb{E}_{R_k}\left[\prod_{j=1, j\ne k}^{K} \mathbb{P}\left(R_j > \left(\frac{B_j P_j}{B_k P_k}\right)^{1/\alpha_j} R_k^{\alpha_k/\alpha_j} | R_k\right)\right] \\ &= \pi\lambda_k \int_0^\infty \exp\left\{-\pi \sum_{j=1}^{K} \lambda_j C_j t^{\alpha_k/\alpha_j}\right\} dt \end{aligned} \tag{2}$$

in which $C_j = \left(\frac{B_j P_j}{B_k P_k}\right)^{2/\alpha_j}$ and the second equality follows by the null probability of a 2-D PPP.

We denote S as the area of the PPP network. The average number of new users that are associated with a typical cell of Tier-k is given by:

$$M_k^{(n)} = \frac{A_k \lambda^{(u)} S}{\lambda_k S} = \pi \lambda^{(u)} \int_0^\infty \exp\left\{ -\pi \sum_{j=1}^{K} \lambda_j C_j t^{\alpha_k/\alpha_j} \right\} dt \tag{3}$$

Furthermore, the Probability Density Function (PDF) of the distance from the user to the serving BS, r_k, is shown in [8]:

$$f_{R_k}(r_k) = \frac{2\pi\lambda_k}{A_k} r_k \exp\left\{ -\pi \sum_{j=1}^{K} \lambda_j C_j r_k^{2\alpha_k/\alpha_j} \right\} \tag{4}$$

3.2 Frequency Reuse Algorithm

We assume that all cells in a given tier use the same FR pattern including a resource allocation technique and a FR factor Δ_k which is used to define the number of cells using the same FR pattern. During the establishment phase, $M_k^{(o)}$ users in each cell are classified into $M_k^{(oc)}$ CCUs and $M_k^{(oe)}$ CEUs. Similarly, the $M_k^{(n)}$ new users are also classified into $M_k^{(ne)}$ CEUs and $M_k^{(nc)}$ CCUs. Correspondingly, N_k available RB in each cell of Tier-k are divided into $N_k^{(c)}$ CC RBs and $N_k^{(e)}$ CE RBs. In the case of the Soft FR, each BS is allowed to use the entire $N_k^{(c)}$ RBs for CCUs and $N_k^{(e)}$ RBs for CEUs while each BS under the Strict FR is only allowed to use $N_k^{(c)}$ RBs for CCUs and $\frac{N_k^{(e)}}{\Delta_k}$ RBs for CEUs. Conventionally, the BS transmits at a low power on the CC RB, called CC power, and at a high power on CE RB, called CE power. The ratio between the CE and CC powers is defined as a transmit ratio. For Tier-k, it is denoted by ϕ_k.

We denote $\theta_k^{(c)}$ and $\theta_k^{(e)}$ as the set of interfering BSs transmitting at the CC and CE powers in Tier-k; $\lambda_k^{(c)}$ and $\lambda_k^{(e)}$ are the densities of BSs in $\theta_k^{(c)}$ and $\theta_k^{(e)}$ in which $\lambda_k^{(c)} = \lambda_k$ and $\lambda_k^{(e)} = \frac{\lambda_k}{\Delta_k}$ under the Strict FR, and $\lambda_k^{(c)} = \frac{\Delta_k - 1}{\Delta_k}\lambda_k$ and $\lambda_k^{(e)} = \frac{1}{\Delta_k}\lambda_k$ under the Soft FR [3].

In an LTE network, each user reports the channel quality periodically or non-periodically [19]. However, in order to reduce the signalling load, the user should not send the measured SINR every timeslot. Conventionally, the reporting interval can be adjusted based on the uplink traffic load [20]. Hence, it is assumed that when new users arrive, the existing users do not report the channel state to the BSs. Thus, the existing CCUs (CEUs) are continuously served as CCUs (CEUs) while each new user is defined as either a CCU or CEU.

We denote $z = (c, e)$ in which $z = c$ and $z = e$ correspond to the CC and CE, respectively. The numbers of CCUs and CEUs per cell in Tier-k are obtained by the following equation:

$$M_k^{(oz)} + M_k^{(nz)} = M_k^{(z)} \tag{5}$$

3.3 Scheduling Algorithm

Each typical user z in Tier-k can be randomly allocated an available RB out of $N_k^{(z)}$ RBs. We define $\tau_k^{(z)}$ as an indicator function that takes value of 1 if the RB b is used at the z cell area of Tier-k and zero otherwise. The expected values of $\tau_k^{(z)}$ is given by:

$$\epsilon_k^{(z)} = \mathbb{E}[\tau_k^{(z)}] = \frac{M_k^{(z)}}{N_k^{(z)}}, \quad \forall 1 \leq k \leq K \tag{6}$$

During the establishment phase when $M_k^{(nz)} = 0$ for both $z = c$ and $z = e$, the allocation ratio for z Area in Tier-k is denoted by $\epsilon_k^{(oz)}$. When the average number of users at a given area of a cell in Tier-k is greater than the average number of allocated RBs, i.e, all RBs are used at the same time, the corresponding allocation ratio is set to 1.

*Strict FR*Since in Strict FR, the CEUs do not share the allocated resource with the CCUs and vice versa, every user only experiences the ICI caused by the BSs transmitting at the same power with that of the serving BS. The ICI of user z is obtained by the following equation:

$$I_{Str}^{(z)} = \sum_{j=1}^{K} \left\{ \sum_{z_c \in \theta_j^{(z)}} \tau_k^{(z)} \tau_j^{(z_c)} \phi_j^{(z)} P_j g_{jy} r_{jy}^{-\alpha_j} \right\} \tag{7}$$

in which $\phi_j^{(c)} = 1, \phi_j^{(e)} = \phi_j$; g_{jy} and r_{jy} are the channel power gain and distance from the user to the interfering BS y in Tier-j, $y \in \{z_c, z_e\}$ correspond to CC and CE Areas.

*Soft FR*In Soft FR, the BS can reuse any RB which means that an RB can be used as a CC RB at a cell and re-used as a CE RB at other cells. Therefore, every user suffers the ICI from the BSs transmitting at the CC and CE powers. The ICI at user z in Tier-k is given by:

$$I_{Sof}^{(z)} = \sum_{j=1}^{K} \left\{ \sum_{z_c \in \theta_j^{(z)}} \tau_k^{(z)} \tau_j^{(z_c)} P_j g_{jz_c} r_{jz_c}^{-\alpha_j} + \sum_{z_e \in \theta_j^{(e)}} \tau_k^{(z)} \tau_j^{(z_e)} \phi_j P_j g_{jz_e} r_{jz_e}^{-\alpha_j} \right\} \tag{8}$$

in which $\theta_j^{(z)}$ is the set of BSs in Tier-j transmitting at z power.

*Signal-to-Interference-plus-Noise Ratio*The instantaneous received SINR of user z from the serving BS in Tier-k is obtained by:

$$SINR_k(\phi_k^{(z)}, r_k) = \frac{\phi_k^{(z)} P_k g_k r_k^{-\alpha_k}}{\sigma_G^2 + I_{FR}^{(z)}} \tag{9}$$

in which $I_{FR}^{(z)}$ is the ICI of the user, $FR = (Sof, Str)$ correspond to Soft FR and Strict FR.

3.4 Number of new CCUs and CEUs

When a user requests communication with the network, it measures the SINR on the data channel and sends the measured SINR to the BS. The user is served as a CCU if its measured SINR is greater than the SINR threshold. Therefore, the probability in which the user in Tier-k is served as the CCU is given by $\mathbb{P}(SINR^{(o)}(1, r_k) > T_k)$. The average number of CCUs in a typical cell in Tier-k is $\frac{\lambda^{(u)}}{\lambda_k}\mathbb{P}(SINR^{(o)}(1, r_k) > T_k)$.

Theorem 1 *(Strict FR) The average number of new CCUs per cell in Tier-k is given by*

$$M^{(nc)}_{Str,k}(T_k) = \pi\lambda^{(u)} \int_0^\infty e^{-\frac{T_k t^{\alpha/2}}{SNR_k} - \pi \sum_{j=1}^K \lambda_j C_j t^{\alpha_k/\alpha_j}\left(1+\epsilon_k^{(oc)} \upsilon_j^{(oc)}(T_k)\right)} dt \tag{10}$$

and the average number of CEUs is

$$\begin{aligned} M^{(ne)}_{Str,k}(T) = & \frac{\pi\lambda^{(u)}}{\lambda_k} \int_0^\infty e^{-\pi \sum_{j=1}^K \lambda_j C_j t^{\frac{\alpha_k}{\alpha_j}}} dt \\ & - \pi\lambda^{(u)} \int_0^\infty e^{-\frac{T_k t^{\alpha/2}}{SNR_k} - \pi \sum_{j=1}^K \lambda_j C_j t^{\alpha_k/\alpha_j}\left(1+\epsilon_k^{(oc)} \upsilon_j^{(oc)}(T_k)\right)} dt \end{aligned} \tag{11}$$

where the symbols are defined in Table 1 and $\upsilon_j^{(oc)}(T_k) = \int_0^1 \frac{\epsilon_j^{(oc)}}{\frac{1}{T_k}\frac{B_j}{B_k} x^{2-\alpha_j/2} + x^2} dx$.

Proof: *Since in Strict FR, the user on a CC RB is only affected by the ICI from the BSs transmitting at the CC power, the probability where the user in Tier-k is defined as the CCU is obtained by using Appendix A with* $\lambda_k^{(c)} = \lambda_k$ *and* $\lambda_k^{(e)} = 0$.

Theorem 2 *(Soft FR) The average numbers of new CCUs and CEUs per cell in Tier-k are given by* $M^{(nc)}_{Sof,k}(T_k)$ *and* $M^{(ne)}_{Sof,k}(T_k)$ *where*

$$M^{(nc)}_{Sof,k}(T_k) = \pi\lambda^{(u)} \int_0^\infty e^{-\frac{T_k r_k^{\alpha_k/2}}{SNR_k} - \pi \sum_{j=1}^K \lambda_j C_j \epsilon_k^{(oc)} \rho_j^{(o)}(T_k) t^{\frac{\alpha_k}{\alpha_j}}} dt \tag{12a}$$

$$M^{(ne)}_{Sof,k}(T_k) = \frac{\pi\lambda^{(u)}}{\lambda_k} \int_0^\infty e^{-\pi \sum_{j=1}^K \lambda_j C_j t^{\frac{\alpha_k}{\alpha_j}}} dt - \pi\lambda^{(u)} \int_0^\infty e^{-\frac{T_k r_k^{\alpha_k/2}}{SNR_k} - \pi \sum_{j=1}^K \lambda_j C_j \epsilon_k^{(oc)} \rho_j^{(o)}(T_k) t^{\frac{\alpha_k}{\alpha_j}}} dt \tag{12b}$$

in which

$$\rho_j^{(o)}(T_k) = \frac{1}{\Delta_j} \int_0^1 \frac{\epsilon_j^{(oe)}}{\frac{1}{T_k}\frac{B_j}{\phi_j B_k} x^{2-\alpha_j/2} + x^2} dx + \frac{\Delta_j - 1}{\Delta_j} \int_0^1 \frac{\epsilon_j^{(oc)}}{\frac{1}{T_k}\frac{B_j}{B_k} x^{2-\alpha_j/2} + x^2} dx \tag{13}$$

Proof: See A.

4 Coverage Probability

4.1 Coverage Probability Definition

The coverage probability of a typical user in Tier-k for a given coverage threshold $\hat{T}_k$ is defined as the probability in which the received SINR is greater than the threshold $\hat{T}_k$.

When the user is defined as the CCU, the BS serves the user on the CC RB at the CC power. Since, the instantaneous channel gain changes from g_k during the establishment phase into g'_k during the communication phase, the new SINR will be denoted by $SINR(1, r_k)$ and obtained from Equation (9). Thus, the coverage probability is defined as:

$$\mathcal{P}_k^{(c)}(T_k, \hat{T}_k) = \mathbb{P}\left(SINR(1, r_k) > \hat{T}_k | SINR^{(o)}(1, r_k) > T_k\right) \tag{14}$$

When the user is defined as the CEU, the BS serves the user on the CE RB at the CE power. The new SINR of the user at the transmission phase is denoted by $SINR(\phi_k, r_k)$. The coverage probability of the CEU can be written as:

$$\mathcal{P}_k^{(e)}(T_k, \hat{T}_k) = \mathbb{P}\left(SINR(\phi_k, r_k) > \hat{T}_k | SINR^{(o)}(1, r_k) < T_k\right) \tag{15}$$

in which T_k and $\hat{T}_k$ are the SINR threshold and coverage threshold for Tier-k; $SINR_k^{(o)}(1, r_k)$ and $SINR_k(\phi_k, r_k)$ are defined as in Equation (9).

The user is under the coverage of the network if it is under the coverage area of any tier. Hence, the average coverage probability of the user in the network is

$$\mathcal{P}_c^{(z)} = \sum_{k=1}^{K} A_k P_k^{(z)}(T_k, \hat{T}_k) \tag{16}$$

in which $z = (c, e)$ and A_k is the probability that the user is connected to Tier-k.

4.2 Coverage Probabilities of CCU and CEU

Theorem 3 *(Strict FR, CCU) The CCU average coverage probability in Tier-k is given by*

$$\mathcal{P}_{Str,k}^{(c)}(T_k, \hat{T}_k) = \frac{\int_0^\infty e^{-\frac{(\hat{T}_k + T_k) t^{\frac{\alpha_k}{2}}}{SNR_k} - \pi \sum_{j=1}^K \lambda_j C_j t^{\alpha_k/\alpha_j} \times \begin{bmatrix} \epsilon_k^{(oc)} \upsilon_j^{(c)}(T_k) + \epsilon_k^{(c)} \upsilon_j^{(c)}(\hat{T}_k) \\ + 1 - \epsilon_k^{(oc)} \epsilon_k^{(c)} \kappa^{(c)}(T_k, \hat{T}_k) \end{bmatrix}} dt}{\int_0^\infty e^{-\frac{T_k t^{\alpha_k/2}}{SNR_k} - \pi \sum_{j=1}^K \lambda_j t^{\alpha_k/\alpha_j} \left(1 + \epsilon_k^{(oc)} \upsilon_j^{(oc)}(\hat{T}_k)\right)} dt} \tag{17}$$

in which $\kappa^{(c)}(T_k, \hat{T}_k) = \int_0^1 \frac{\epsilon_j^{(oc)} \epsilon_j^{(c)}}{x^2 \left(\frac{1}{T_k} \frac{B_j}{B_k} x^{-\alpha_j/2} + 1\right)\left(\frac{1}{\hat{T}_k} \frac{B_j}{B_k} x^{-\alpha_j/2} + 1\right)} dx$ *and*
$\upsilon_j^{(c)}(\hat{T}_k) = \int_0^1 \frac{\epsilon_j^{(c)}}{\frac{1}{\hat{T}_k} \frac{B_j}{B_k} x^{2-\alpha_j/2} + x^2} dx.$

Proof: See B

One of the differences between the CCU and CEU is that the density of the interfering BSs in Tier-j of the CCU is λ_j while that of the CEU is λ_j / Δ_j. Furthermore, although the CEU is served with high transmit power, the transmit power of the serving and interfering BSs of the CEU are the same. Thus, the average coverage probability of the CUE is given by Theorem 4.

Theorem 4 *(Strict FR, CEU) The average coverage probability of the CEU in Tier-k is given by*

$$\mathcal{P}^{(e)}_{Str,k}(T_k,\hat{T}_k)=\frac{\pi\lambda_k\int_0^\infty\left[\begin{array}{l} e^{-\frac{\hat{T}_k t^{\alpha_k/2}}{\phi_k SNR_k}-\pi\sum_{j=1}^K\lambda_j t^{\alpha_k/\alpha_j}\left(1+\epsilon_k^{(e)}\upsilon_j^{(e)}(\hat{T}_k)\right)}-e^{-\frac{t^{\alpha_k/2}}{SNR_k}\left(\frac{\hat{T}_k}{\phi_k}+T_k\right)} \\ \times e^{-\pi\sum_{j=1}^K\lambda_j C_j t^{\alpha_k/\alpha_j}\times\left[\begin{array}{r}\epsilon_k^{(oc)}\upsilon_j^{(oc)}(T_k)+\frac{1}{\Delta_j}\epsilon_k^{(e)}\upsilon_j^{(e)}(\hat{T}_k)\\ +1-\frac{1}{\Delta_j}\epsilon_k^{(oc)}\epsilon_k^{(e)}\kappa^{(e)}(T_k,\hat{T}_k)\end{array}\right]}\end{array}\right]dt}{1-\int_0^\infty e^{-\frac{T_k t^{\alpha_k/2}}{SNR_k}-\pi\sum_{j=1}^K\lambda_j t^{\alpha_k/\alpha_j}\left(1+\epsilon_k^{(oc)}\upsilon_j^{(oc)}(\hat{T}_k)\right)}dt} \tag{18}$$

in which $\kappa^{(e)}(T_k,\hat{T}_k)=\int_0^1\frac{\epsilon_j^{(oc)}\epsilon_j^{(e)}}{x^2\left(\frac{1}{T_k}\frac{B_j}{B_k}x^{-\alpha_j/2}+1\right)\left(\frac{1}{\hat{T}_k}\frac{B_j}{B_k}x^{-\alpha_j/2}+1\right)}$ *and*
$\upsilon_j^{(e)}(\hat{T}_k)=\int_0^1\frac{\epsilon_j^{(e)}}{\frac{1}{\hat{T}_k}\frac{\phi_k}{\phi_j}\frac{B_j}{B_k}x^{2-\alpha_j/2}+x^2}dx$ *and*

Proof:See C

Theorem 5 *(Soft FR, CCU) The average coverage probability of a typical CCU associated with Tier-k in a Soft FR network when the user connects to Tier-k is given by*

$$\mathcal{P}^{(c)}_{Sof,k}(1,\hat{T}_k)=\frac{\int_0^\infty e^{-\frac{t^{\alpha_k/2}}{SNR_k}(\hat{T}_k+T_k)-\pi\lambda_j C_j t^{\alpha_k/\alpha_j}\times\left[\begin{array}{r}\epsilon_k^{(oc)}\rho_j^{(o)}(T_k)+\epsilon_k^{(c)}\rho_j(\hat{T}_k)\\ +1-\epsilon_k^{(oc)}\epsilon_k^{(c)}\Gamma^{(c)}(T_k,\hat{T}_k)\end{array}\right]}dt}{\int_0^\infty e^{-\frac{T_k t^{\alpha_k/2}}{SNR_k}-\pi\sum\limits_{j=1}^K\lambda_j C_j t^{\alpha_k/\alpha_j}\left(1+\epsilon_k^{(oc)}\rho_j^{(o)}(T_k)\right)}dt} \tag{19}$$

where $\rho_j(T_k)=\frac{1}{\Delta_j}\int_0^\infty\frac{\epsilon_j^{(e)}}{\frac{1}{T_k}\frac{B_j}{\phi_j B_k}x^{2-\alpha_j/2}+x^2}dx+\frac{\Delta_j-1}{\Delta_j}\int_0^\infty\frac{\epsilon_j^{(c)}}{\frac{1}{T_k}\frac{B_j}{B_k}x^{2-\alpha_j/2}+x^2}dx$*; and*
$\Gamma_j(T_k,\hat{T}_k)=\frac{\Delta_j-1}{\Delta_j}\int_0^1\frac{\epsilon_j^{(oc)}\epsilon_j^{(c)}}{x^2\left(1+\frac{1}{T_k}\frac{B_j}{B_k}x^{-\alpha_j/2}\right)\left(1+\frac{1}{\hat{T}_k}\frac{B_j}{B_k}x^{-\alpha_j/2}\right)}dx+\frac{1}{\Delta_j}\int_0^1\frac{\epsilon_j^{(oe)}\epsilon_j^{(e)}}{x^2\left(1+\frac{1}{T_k}\frac{B_j}{\phi_j B_k}x^{-\alpha_j/2}\right)\left(1+\frac{1}{\hat{T}_k}\frac{B_j}{\phi_j B_k}x^{-\alpha_j/2}\right)}dx.$

Proof: See D.

Theorem 6 *(Soft FR, CEU) The average coverage probability of the CEU in Tier-k in a Soft FR is*

$$\mathcal{P}^{(e)}_{Str,k}(T_k,\hat{T}_k)=\frac{\pi\lambda_k\int_0^\infty\left[\begin{array}{l} e^{-\frac{\hat{T}_k t^{\alpha_k/2}}{\phi_k SNR_k}-\pi\sum\limits_{j=1}^K\lambda_j C_j t^{\alpha_k/\alpha_j}\left(1+\epsilon_k^{(e)}\rho_j(\frac{\hat{T}_k}{\phi_k})\right)}-e^{-\frac{t^{\alpha_k/2}}{SNR_k}\left(\frac{\hat{T}_k}{\phi_k}+T_k\right)} \\ \times e^{-\pi\sum_{j=1}^K\lambda_j C_j t^{\alpha_k/\alpha_j}\times\left[\begin{array}{r}\epsilon^{(oc)}\rho_j^{(o)}(T_k)+\epsilon^{(e)}\rho_j(\frac{\hat{T}_k}{\phi_k})\\ +1-\epsilon_k^{(oc)}\epsilon_k^{(e)}\Gamma_j(\frac{\hat{T}_k)}{\phi_k}\end{array}\right]}\end{array}\right]dt}{1-\pi\lambda_k\int_0^\infty e^{-\frac{T_k t^{\alpha_k/2}}{SNR_k}-\pi\sum\limits_{j=1}^K\lambda_j C_j t^{\alpha_k/\alpha_j}\left(1+\epsilon_k^{(oc)}\rho_j^{(o)}(T_k)\right)}dt} \tag{20}$$

in which $\rho_j^{(o)}(T_k)$*,* $\rho_j(T_k)$ *and* $\Gamma_j(T_k,\hat{T}_k)$ *are defined in Theorem 2 and Theorem 5 respectively.*

***Proof**: The coverage probability expression of the CEU under the Soft FR is*

$$\begin{aligned}\mathcal{P}_c^{(c)}(T_k,\hat{T}_k)&=\frac{\mathbb{P}\left(\frac{\phi_k P_k g'_k r_k^{-\alpha_k}}{\sigma_G^2+I_{Soft}^{(e)}}>\hat{T}_k,\frac{P_k g_k r_k^{-\alpha_k}}{\sigma_G^2+I_{Soft}^{(oc)}}<T_k\right)}{\mathbb{P}\left(\frac{P_k g_k r_k^{-\alpha_k}}{\sigma_G^2+I_{Soft}^{(oc)}}<T_k\right)}\\ &=\frac{\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2}\mathbb{E}\left[e^{-\frac{\hat{T}_k r_k^{\alpha_k}}{\phi_k P_k}\left(\sigma_G^2+I_{Soft}^{(e)}\right)}\left(1-e^{-\frac{T_k r_k^{\alpha_k}}{P_k}\left(\sigma_G^2+I_{Soft}^{(oc)}\right)}\right)\right]dr_k}{1-\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2}\mathbb{E}\left[e^{-\frac{T_k}{P_k}\left(\sigma_G^2+I_{Soft}^{(oc)}\right)}\right]dr_k}\end{aligned} \tag{21}$$

In (21), the numerator can be saperated into two expected expressions which are evaluated using the results in Appendix A and Appendix B with coverage threshold $\frac{\hat{T}_k}{\phi_k}$ *and allocation ratio for the second phase of* $\epsilon_k^{(e)}$*. The denominator can be obtained by Appendix A. Therefore, the average coverage probability of CEU is given by (20).*

5 Average Cell Data Rate

In this section, the two phase operation of FR algorithms are deployed to derive the average capacity of CCU and CEU. In addtion, the total average data rate of all users in a typcial cell which called average cell data rate is computed.

5.1 Average Capacity of CCU and CEU

The average data rate of the user with the received signal $SINR$ is given by the Shannon Theorem, i.e, $C = \mathbb{E}[\,ln(1 + SINR)]$ in which the expectation is taken over the SINR distribution. In the FR network, the CCU in Tier-k experiences a received SINR at $SINR(1, \phi_k)$ during the communication phase if the measured SINR during the establishment phase is $SINR^{(o)}(1, r_k) > T_k$. Hence, the average data rate of the CCU is obtained by [5]:

$$\begin{aligned} C^{(c)}_{FR,k}(T_k, 1) &= \mathbb{E}\left(\ln\left(SINR(1, r_k) + 1\right)|SINR^{(o)}(1, r_k) > T_k\right) \\ &= \int_0^\infty \mathbb{P}\left(\ln\left(SINR(1, r_k) + 1\right) > \gamma|SINR^{(o)}(1, r_k) > T_k\right) d\gamma \\ &= \int_0^\infty \frac{\mathbb{P}\left(SINR(1, r_k) > e^\gamma - 1, SINR^{(o)}(1, r_k) > T_k\right)}{\mathbb{P}\left(SINR^{(o)}(1, r_k) > T_k\right)} d\gamma \end{aligned} \tag{22}$$

Employing a change of variable $t = e^\gamma - 1$, Equation 22 is obtained by

$$\begin{aligned} C^{(c)}_{FR,k}(T_k, 1) &= \int_0^\infty \frac{1}{t+1} \frac{\mathbb{P}\left(SINR(1, r_k) > t, SINR^{(o)}(1, r_k) > T_k\right)}{\mathbb{P}\left(SINR^{(o)}(1, r_k) > T_k\right)} dt \\ &= \int_0^\infty \frac{1}{t+1} \mathcal{P}^{(c)}_{FR,k}(T_k, \hat{T}_k) dt \end{aligned} \tag{23}$$

in which $\mathcal{P}^{(c)}_{FR,k}(T_k, \hat{T}_k)$ is defined by Equation 17 in the case of Strict FR and Equation 19 in the case of Soft FR.

Similarly, the average data rate of the CEU in this case is obtained by [5]:

$$C^{(e)}_k(T_k, \phi_k) = \int_0^\infty \frac{1}{t+1} \mathcal{P}^{(e)}_{FR,k}(T_k, \hat{T}_k) dt \tag{24}$$

in which $\mathcal{P}^{(e)}_{FR,k}(T_k, \hat{T}_k)$ is defined by Equation 18 in the case of Strict FR and Equation 20 in the case of Soft FR.

5.2 Average Cell Capacity

The average cell data rate in Tier-k is defined as the sum of the data rates of all users, and can be obtained as:

$$R_k(T_k) = M^{(e)}_k C^{(e)}_k(T_k, \phi_k) + M^{(c)}_k C^{(c)}_k(T_k, 1) \tag{25}$$

in which $M_k^{(e)}$ and $M_k^{(c)}$ are the numbers of CEUs and CCUs connected to a typical cell in Tier-k and are given in Section 1 and 2; $C_{FR,k}^{(e)}(T_k, \phi_k)$ and $C_{FR,k}^{(c)}(1, \phi_k)$ are average data rates of CEU and CCU in Tier-k which are given in (23) and (24).

6 Simulation Results and Discussion

In this section, the numerical results for the 2-tier network are analysed. Conventionally, the macro BSs with a higher transmit power are distributed with a lower density than pico BSs with a lower transmit power. The macro BSs are distributed with a density of $\lambda_1 = 0.05$ BS/km^2 which is 10 times less than that of the pico BSs $\lambda_2 = 0.5$ BS/km^2. Since the average number of existing users during the initial network state is used to estimate the ICI when the new users arrive and do not affect the trends of the network performance, it can be assumed to be constant. In our simulation study, 3 CCUs and 6 CEUs are chosen for each cell in Tier-1, while 3 CCUs and 5 CEUs are selected for each cell in Tier-2. In a practical network, the BSs can select some of the users with high SINRs to establish the initial state of the network. Thee number of new users requesting communication with the network is assumed to be a Poisson RV with mean $\lambda^{(u)} = 5$ $user/km^2$.

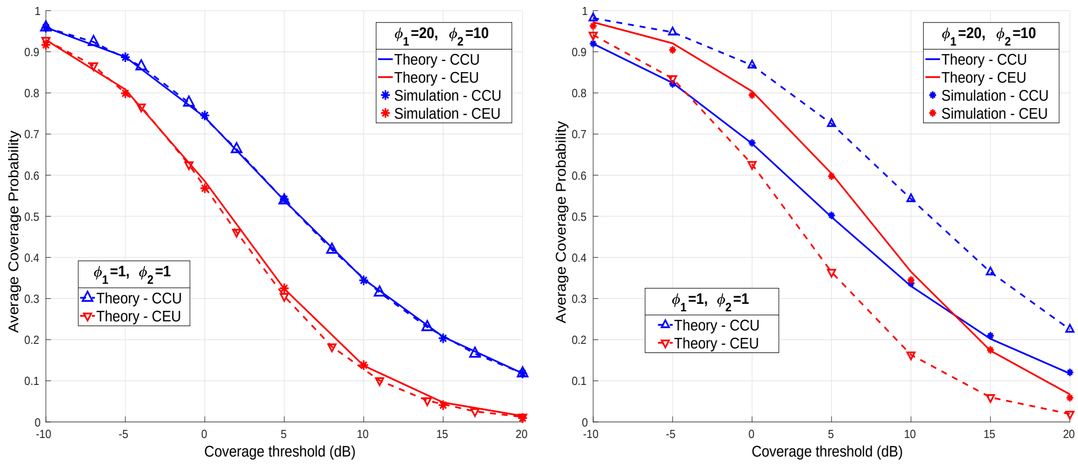
Each cell in each tier of the network is allowed to share 75 RBs corresponding to 15 MHz and utilises the FR algorithm with a reuse factor of 3. In Soft FR, the 75 RBs are divided in 3 groups of 25 RBs, two of the groups are assigned to CC Area and the last group is assigned to CE Area. In Strict FR, the 75 RBs are partitioned into one common group of 30 RBs that is assigned to CC Area of each cell and 3 private groups of 15 RBs. Each private group is assigned to CE Area of each cell in a group of 3 cells. The simulation parameters are summarised in Table 2.

Table 2 Analytical and simulation parameters

Parameters	**Value**
Number of tiers	$K = 2$
Density of BSs	$\lambda_1 = 0.05$, $\lambda_2 = 0.5$ BS/km^2
Frequency Reuse factor	$\Delta_1 = 3$, $\Delta_2 = 3$
Density of new users	$\lambda^{(u)} = 5$ $user/km^2$
Transmit power	$P_1 = 53$ dBm, $P_2 = 33$ dBm
Thermal noise	-99 dBm (corresponding to 15 MHz bandwidth)
Transmit power ratio	$\phi_1 = 20$, $\phi_2 = 10$
Number of RBs per cell - Soft FR - Strict FR	75 50 CC RBs; 25 CE RBs 30 CC RBs; 15 CE RBs
Network area	$S = 100$ km^2

6.1 Validation of the proposed model

In Figure 1, the average coverage probabilities of the CCU and CEU in a 2-tier PPP network are simulated with an assumption that the coverage thresholds are the same at any tier. The solid lines represent the analytical results which visually match with the stars representing the simulation results. The dashed lines with triangles represent



(a) Strict FR (b) Soft FR

Fig. 1 Comparison between theoretical and simulation results of CCU and CEU average coverage probabilities

the average coverage probability of the CEU if it is served with the CC power, i.e, $\phi_1 = \phi_2 = 1$. It is noted that there are 4 theoretical curves and 2 simulation curves in both Figures 1(a) and 1(b).

Since, the CEU under the Strict FR is only affected by the ICI from the BSs which transmit on CE RBs, an increase in the serving power of the CEU also increases the power of the interfering BSs. Meanwhile, the CEU under the Soft FR experiences the ICI from the BSs transmitting at the CC and CE powers in which the density of the BSs transmitting at the CC power in Tier-j is $(\Delta_j - 1)$ times that of the BSs transmitting at the CE power. Hence, using higher transmit power for the CEU under the Soft FR leads to better efficiency than that under the Strict FR. From the analytical results in Figure 1, it is observed that a high transmit power increases the average coverage probability of the CEU significantly under the Soft FR, but only marginally under the Strict FR. For example, when the coverage threshold is set to $5dB$, the average coverage probability of the CEU under the Soft FR increases by around 92.6 % to approximately 0.5778 when $(\phi_1, \phi_2) = (20, 10)$, compared to 0.3 when $(\phi_1, \phi_2) = (1, 1)$ and. On other hand, under the Strict FR, the CEU average coverage probability has a 6.31% improvement, from 0.3057 to 0.325.

It is noted that the use of a high transmit power to serve CEUs leads to an increase in the ICI of the CEU under the Strict FR and of both the CCU and CEU under the Soft FR. Therefore, an increase in the transmit power on the CE RB does not affect the CCU in the case of Strict FR but can reduce the performance of the CCU under the Soft FR. For example, when the coverage threshold is 5 dB and the transmit ratios (ϕ_1, ϕ_2) increase from $(1, 1)$ to $(20, 10)$, the average coverage probability of the CCU reduces by 30.73% from 0.7251 to 0.5023. Hence, it can be said that with Soft FR, the CCU performance is sacrificed to improve the CEU performance.

6.2 Comparisons between the proposed model and 3GPP model

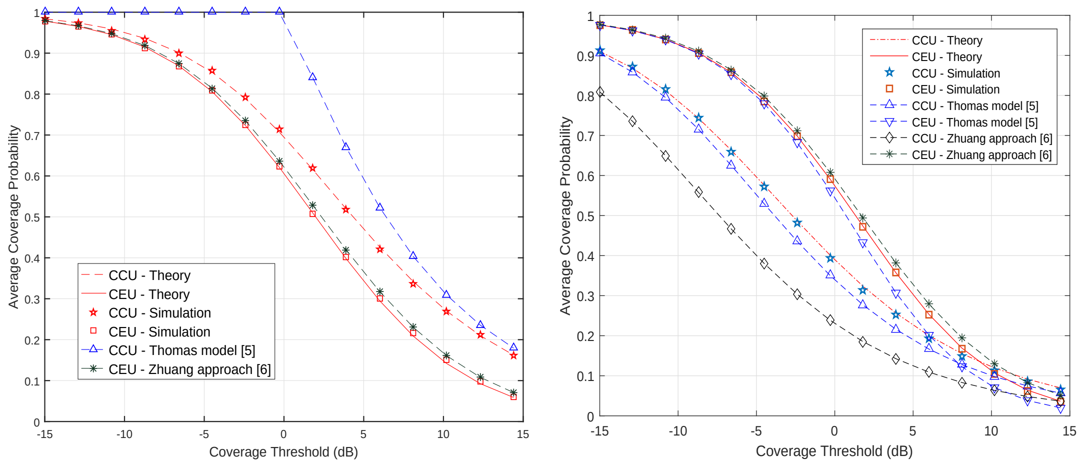
In this section, the 3GPP and proposed models are compared in terms of performance metrics. We follow the 3GPP recommendations to set up the 3GPP network scenarios using FR, in which during the establishment phase the BS measures SINR on the control channel for user classification purpose. Since, each control channel is affected interference from all BSs and all BSs transmit on the control channels at the same power. The analytical results of 3GPP model can be obtained by substituting $\epsilon_j^{(oc)} = \epsilon_j^{(oe)} = 1$ in Equation 17 to Equation 20 with the following notes

– In the case of CEU under Strict FR, the density of interfering BSs during the establishment phase is λ_j instead of λ_j/Δ_j.
– In the case of Soft FR, the transmit power of BSs in both $\theta_{Sof}^{(c)}$ and $\theta_{Sof}^{(e)}$ during the establishment phase are the same at P_j.

The comparison in terms of network performance such as the number of CCU and CEU, user data rate are taken when the network parameters are selected at their optimal values.

6.2.1 Comparison between analytical approaches

We compare our analytical results with the well-known results from [5] and [6] in terms of average coverage probability in the case of a single-tier network ($K = 1$). In Figure 2(a), the performance of CCU under the proposed two-phase operation is compared to the corresponding result in [5], while in Figure 2(b), we compare the proposed analytical approach by separately evaluating interference generated from BSs transmitting on CC and CE RBs to the use of the constant coefficient to evaluate network interference in [5,6].



(a) Strict FR (b) Soft FR

Fig. 2 Comparison the analytical results between the proposed approach and approaches in [5] and [6]

As shown in Fig. 2, the solid lines representing the analytical results perfectly match with the points representing the simulation results but have gaps with the lines which are plotted from the corresponding results in [5] and [6].

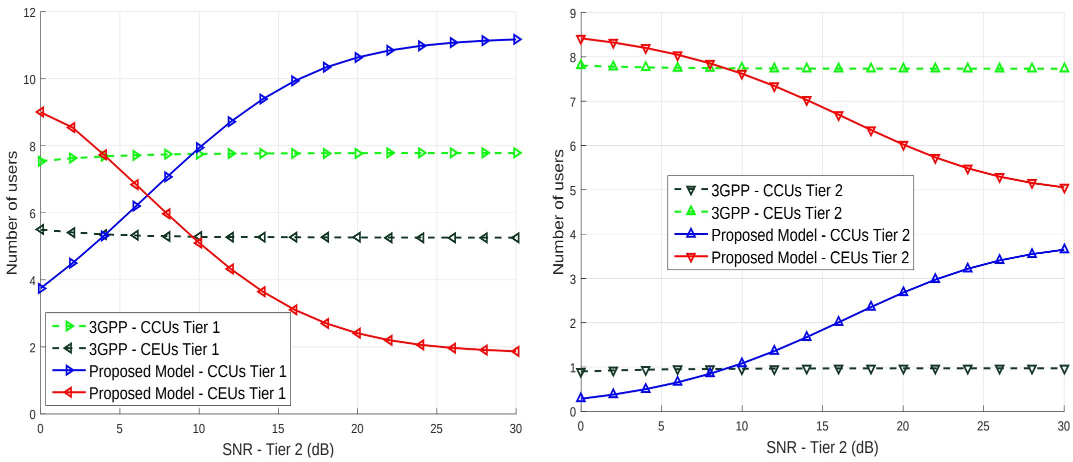
Discussion on the Results in [5]

– In the case of Strict FR: Since [5] assumed that the user transmits the signal for user classification purpose and data at the same time, the user was under the network coverage if the received SINR is greater than both SINR threshold T and coverage threshold $\hat{T}$. Thus, the CCU average coverage probability, $\mathcal{P}_c$, was defined as $\mathbb{P}(SINR > T | SINR > \hat{T})$. Therefore, $\mathcal{P}_c = 1$ if $\hat{T} > T$.

– In the case of Soft FR: In [5], the set of interfering BSs transmitting on CC and CE RBs, $\theta_{Sof}^{(c)}$ and $\theta_{Sof}^{(e)}$, were consolidated by a constant coefficient $\zeta = (\Delta - 1 + \phi)/\Delta$. Thus, the network interference in Equation 8 is given by $I = \sum_{j \in \theta_{Sof}} \zeta P_j^{(c)} g_{jz} r_{jz}^{-\alpha}$ in which $\theta_{Sof}^{(c)} \bigcup \theta_{Sof}^{(e)} = \theta_{Sof}$. In other words, $\theta_{Sof}^{(c)}$ of BSs with transmit power P and density $\frac{\Delta-1}{\Delta}\lambda$ was replaced by θ_{Sof} of BSs with transmit power $\frac{\Delta-1}{\Delta}P$ and density λ; $\theta_{Sof}^{(e)}$ of BSs with transmit power ϕP and density $\frac{1}{\Delta}\lambda$ was considered as equivalent to θ_{Sof} of BSs with transmit power $\frac{\phi}{\Delta}P$ and density λ. However, since two independent sets $\theta_{Sof}^{(c)}$ and $\theta_{Sof}^{(e)}$ are subsets of θ_{Sof}, use of equivalent sets to represent $\theta_{Sof}^{(c)}$ and $\theta_{Sof}^{(e)}$ are not feasible.
In our approach, instead of using the coefficient ζ, $\theta_{Sof}^{(c)}$ and $\theta_{Sof}^{(e)}$ are evaluated separately, hence the analytical results perfectly match with the simulation results. Meanwhile, there are gaps between the simulation results and corresponding results in [5].

Discussion on the Results in [6]:In [6], it was assumed that the interference between the establishment phase and communication phase are independent, thus the joint probability in the numerator of Equation 32 was evaluated as $\mathbb{P}(SINR^{(o)} < T, SINR^{(e)} > \hat{T}|r) = \mathbb{P}(SINR^{(o)} < T|r)\mathbb{P}(SINR^{(e)} > \hat{T}|r)$. However, in downlink cellular networks, the user during both establishment phase and communication phase is experienced interference from the same BSs, thus the interference during both phases are functions of the distance from the user to adjacent BSs. Consequently, $SINR^{(o)}$ and $SINR^{(e)}$ are correlated random variables. As a results, there are also gaps between the results in [6] and simulation results.

6.2.2 Comparison between the numbers of CCUs

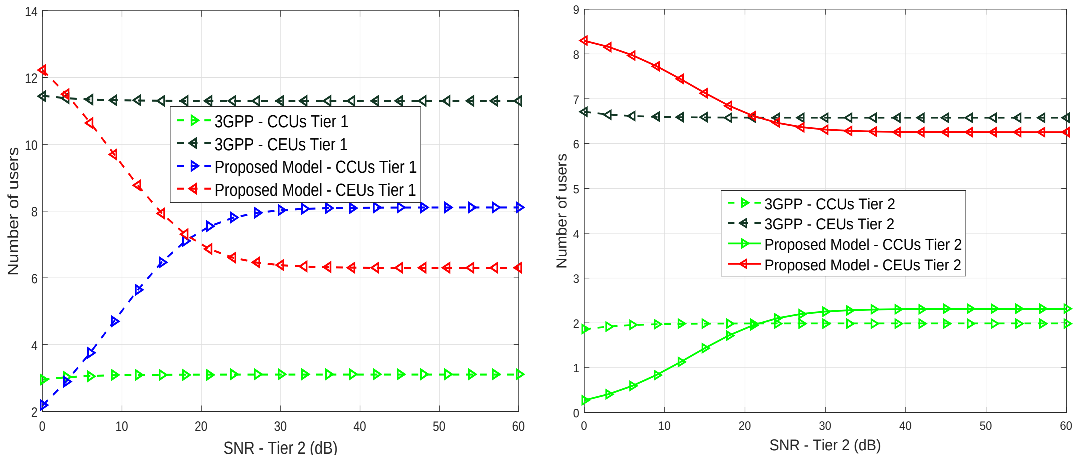


(a) Strict FR, Tier-1 (b) Strict FR, Tier-2

Fig. 3 *Strict FR*: Comparison between number of CCUs and CEUs

The downlink SINR on the control channel of the user associated with Tier-k in the 3GPP model is given by

$$SINR_{3GPP,k} = \frac{g_k r_k^{-\alpha_k}}{\sum\limits_{j=1}^{K} \sum\limits_{y \in \theta_j \setminus \{k\}} \frac{P_j}{P_k} g_{jy} r_{jy}^{-\alpha_j} + 1/SNR_k} \tag{26}$$



(a) Soft FR, Tier-1 (b) Soft FR, Tier-2

Fig. 4 *Strict FR*: Comparison between number of CCUs and CEUs

When the average number of interfering BSs is large enough, $\sum_{j=1}^{K} \sum_{y\in\theta_j\setminus\{k\}} \frac{P_j}{P_k} g_{jy} r_{jy}^{-\alpha_j} >> 1/SNR_k$, hence the measured SINR in Equation 26 and consequently the average number of CCUs depend on $\sum_{j=1}^{K} \sum_{y\in\theta_j\setminus\{k\}} \frac{P_j}{P_k} g_{jy} r_{jy}^{-\alpha_j}$ rather than SNR_k. Therefore, an increase of SNR makes a very small change in the average number of CCUs and CEUs for both Tier-1 and Tier-2 under both Strict FR and Soft FR as shown in Figures 3 and 4.

In our proposed model, the measured SINR for user classification purposed in Equation 9 is a function of the allocation ratios during the establishment phase, which are $\epsilon_1^{(oc)} = 0.1$, $\epsilon_1^{(oe)} = 0.4$, $\epsilon_2^{(oc)} = 0.1$, and $\epsilon_2^{(oe)} = 0.133$. Thus, the interference of the measured SINR is very small compared to that in the case of 3GPP. Hence, an increase in SNR can result in a significant rise of interference, SINR, and consequently number of CCUs as shown in Figures 3 and 4. Take Tier-1 under Strict FR for example, when SNR increases by 8 dB from 0 dB to 8 dB, the average number of CCUs rises by 88.83% from 3.751 to 7.083.

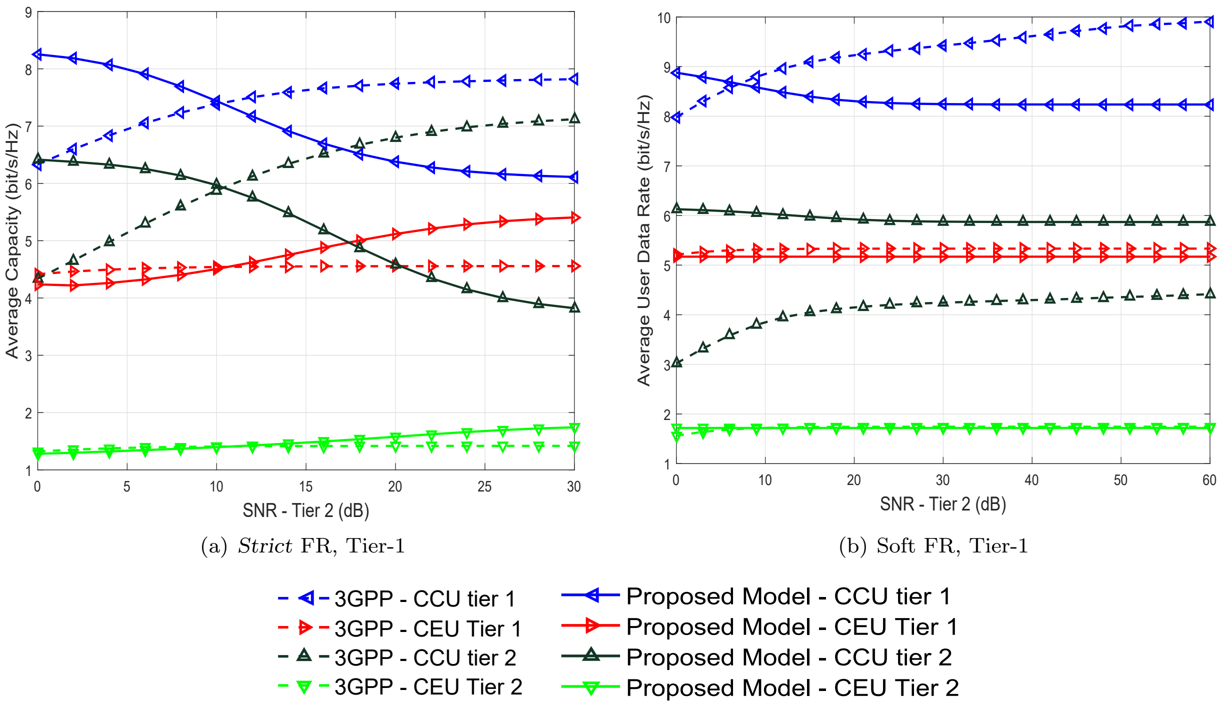
In the cellular network, every BS transmits on both data channel and control channel. In this paper, we focus on the transmit power of the BS in Tier-k on the data channel which is given by

$$\overline{P} = P_k M_k^{(c)} + \phi_k P_k M_k^{(e)} \tag{27}$$

in which $M_k^{(c)}$ and $M_k^{(e)}$ are the average number of CCUs and CEUs, P_k and $\phi_k P_k$ are the serving power of CCU and CEU.

As seen from Equation 27 and the previous discussion, compared to 3GPP model, our proposed model can save a significant amount of the power on data channel by reducing the average number of CEUs. In the case of Soft FR and $SNR = 30$ dB, the transmit powers on data channel of the proposed model are $135.607P_1$ and $128.51P_2$ of a BS in Tier-1 and Tier-2 respectively while those are $229.01P_1$ and $133.57P_2$ in the case of the 3GPP model. Hence, it can be said that the BS in Tier-1 and Tier-2 of our proposed model can save upto 40.79% and 3.8% power consumption on the data channel compared to 3GPP model.

6.2.3 Comparison between the user data rates



(a) *Strict* FR, Tier-1 (b) Soft FR, Tier-1

Fig. 5 *Tier 1*: Comparison between Average User Data Rates

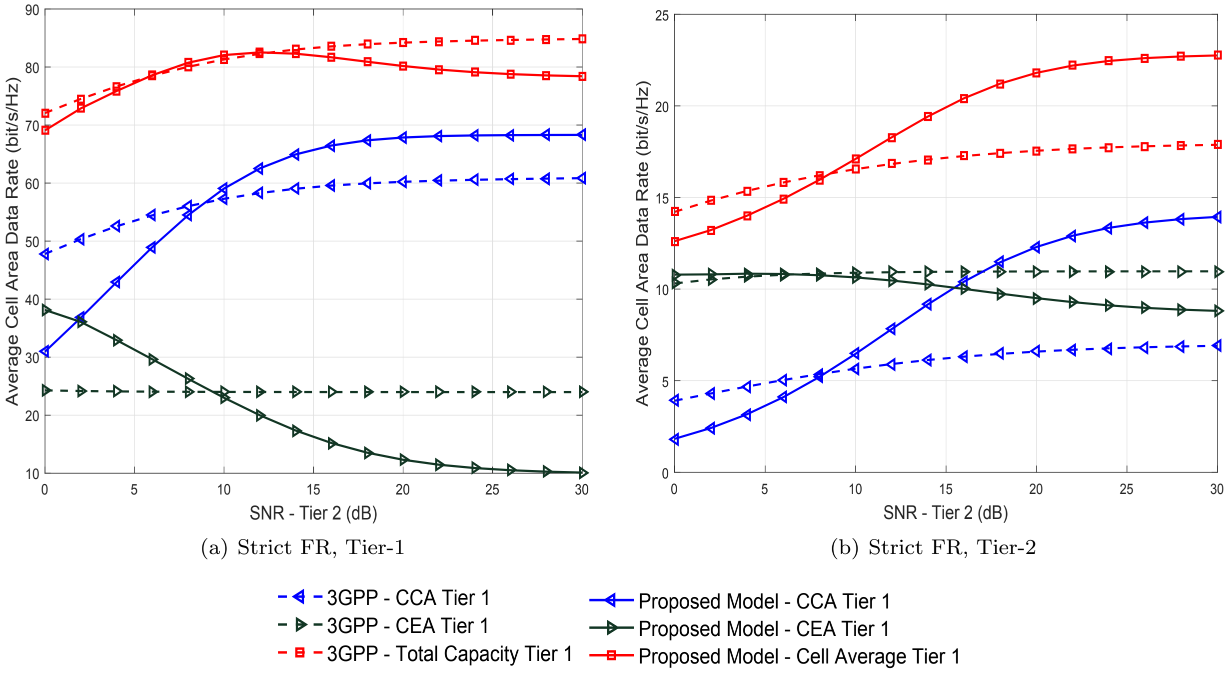
In 3GPP model, since the changes of SNR has a small effect on the user classification probability, an increase of SNR can improve the SINR which reflect in a rise of both CCU and CEU average data rates as shown in Figure 5(a) and 5(b). However, the average data rate of the CCU increase at a significantly lower rate than that of the CEU, since the CEU is always served at a higher transmit power than the CCU. It is observed from Figure 4(a), when SNR increases from 0 dB to 8 dB, the average data rate of the CCU increases by 10.21% from 7.979 to 8.794 while that of the CEU rises by 2.84% from 5.17 to 5.317.

In the proposed model, when SNR increases under Strict FR, the average number of CCUs and consequently interference of the CCU increase while the average number of CEUs and interference power originating from BSs transmitting on the same RB at the CE power reduce. As a result, the average data rate of the CCU decreases while that of the CEU increases as shown in Figure 5(b).

Under Soft FR, it is clear that an increase in the average number of CCUs is equivalent to a decline in the average number of CEU. Since, the density of BSs in $\theta^{(c)}_{Soft}$ is $\Delta_j - 1$ times than that $\theta^{(e)}_{Soft}$, growth in interference originating from $\theta^{(e)}_{Soft}$ can counterbalance a decline in interference originating from $\theta^{(c)}_{Soft}$ though BSs in $\theta^{(e)}_{Soft}$ transmit at higher powers than those in $\theta^{(c)}_{Soft}$. It is noticed that the CEU is served at high transmit power, thus a change of SNR does not significantly affect on the downlink SINR. Therefore, as shown in Figure 5(b), the average data rate of the CCU rises moderately while that of the CEU is unlikely to change.

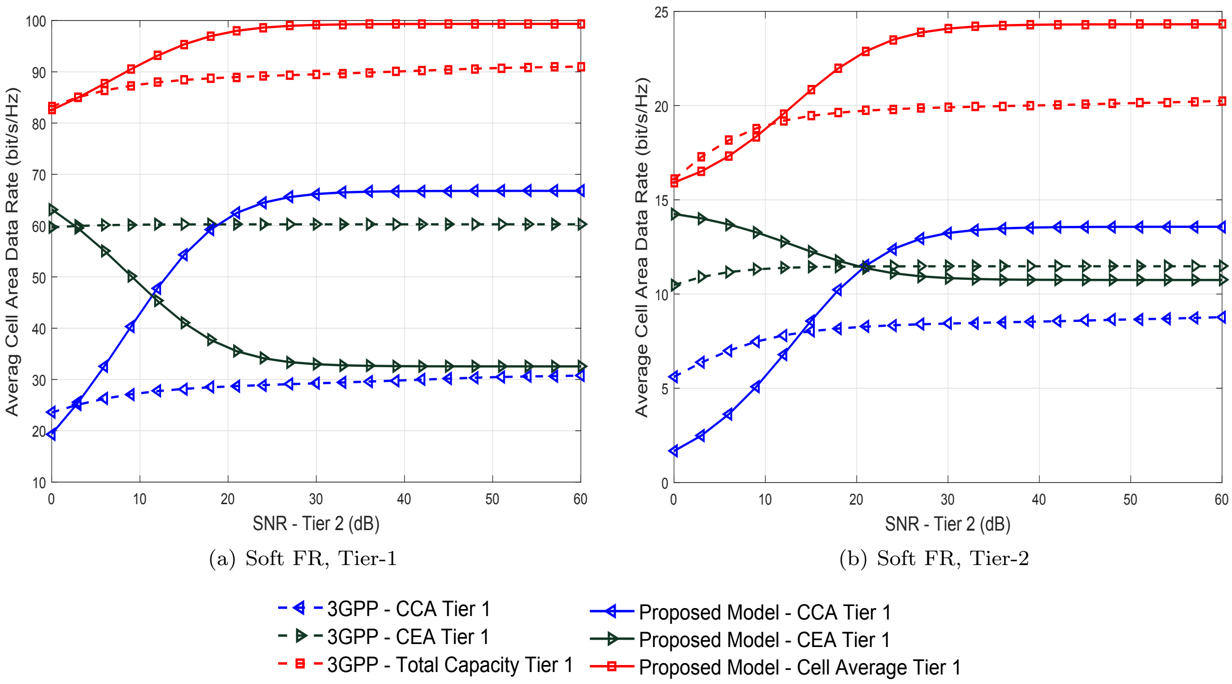
6.2.4 Comparison between Cell Area and Network Data Rates

In 3GPP model, since the average number of CCUs and CEUs are unlikely to change while the average data rate of both CCU and CEU increase with SNR, the average data rate of every cell area, which is defined as the product of the average data of a typical user and number of users in the corresponding cell area, increases with SNR.



(a) Strict FR, Tier-1 (b) Strict FR, Tier-2

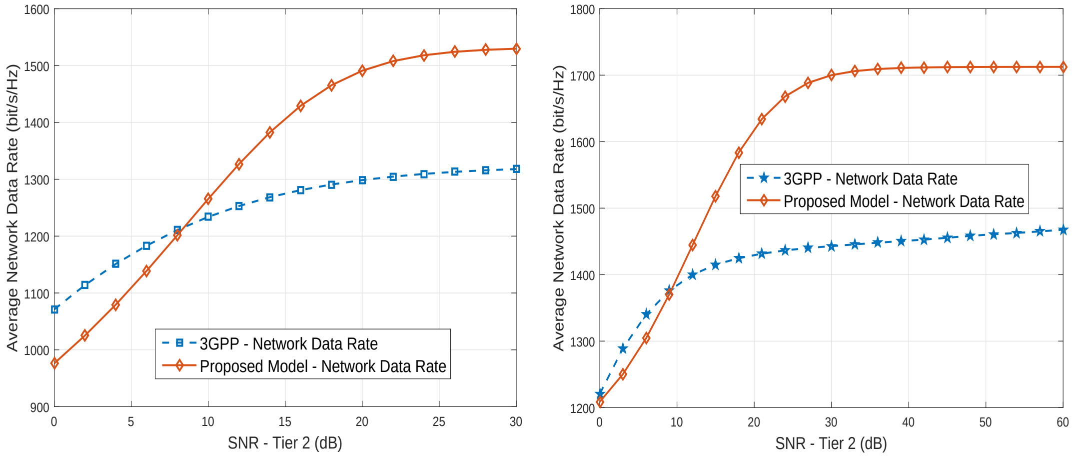
Fig. 6 *Strict FR*: Comparison between Performance of Cell Areas



(a) Soft FR, Tier-1 (b) Soft FR, Tier-2

Fig. 7 *Strict FR*: Comparison between Performance of Cell Areas

In our proposed model, variance of the average number of CCUs and CEUs results in differences between trends of cell area performance as shown in Figures 6 and 7. Although, CCU average data rate reduces when SNR increases, the increase in the average number of CCUs leads to growth in average cell data rate of CC Area under both Strict FR and Soft FR. By contrast, the average number of CEUs reduces while the average data rate of CEU slightly changes, the average data rate of the CE Area under both Strict FR and Soft FR decline when SNR increases.



(a) Strict FR (b) Soft FR

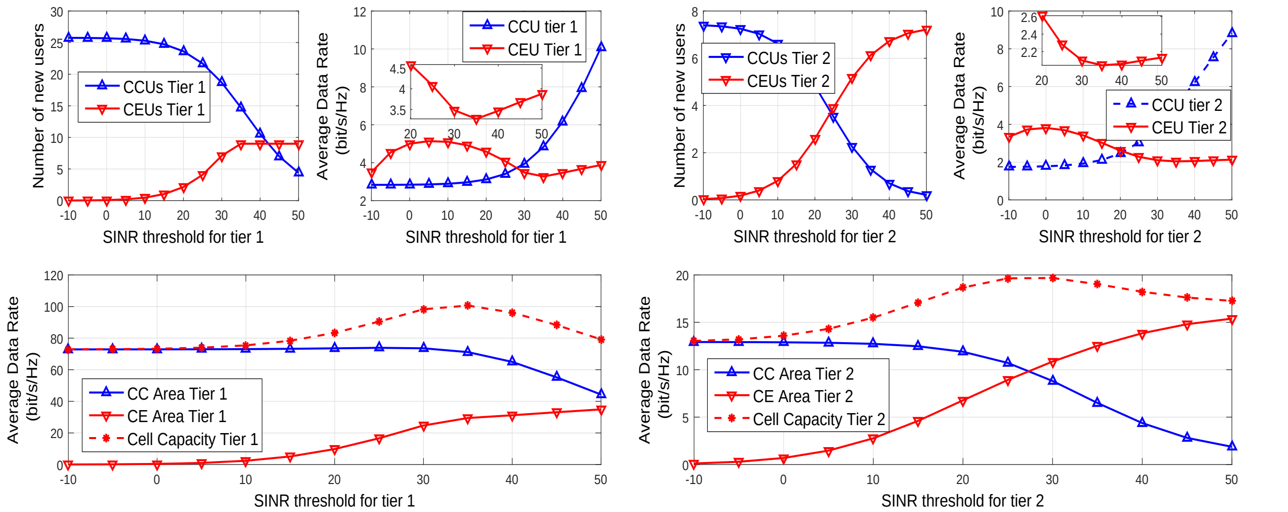
Fig. 8 Average Network Data Rate Comparison

It is observed from Figures 8(a) and 8(b) that when SNR is large enough, such as $SNR > 8$ dB in the case of Strict FR and $SNR > 10$ dB in the case of Soft FR, the average network data rates of the proposed model are significantly greater than those of the 3GPP model. For example, in the case of $SNR = 30$ dB, the proposed model under Strict FR and Soft FR achieve the average network data rates of 1530 (bit/s/Hz) and 1700 (bit/s/Hz) respectively, which are 16.08% and 18.63% greater than the 3GPP models.

6.3 SINR Threshold

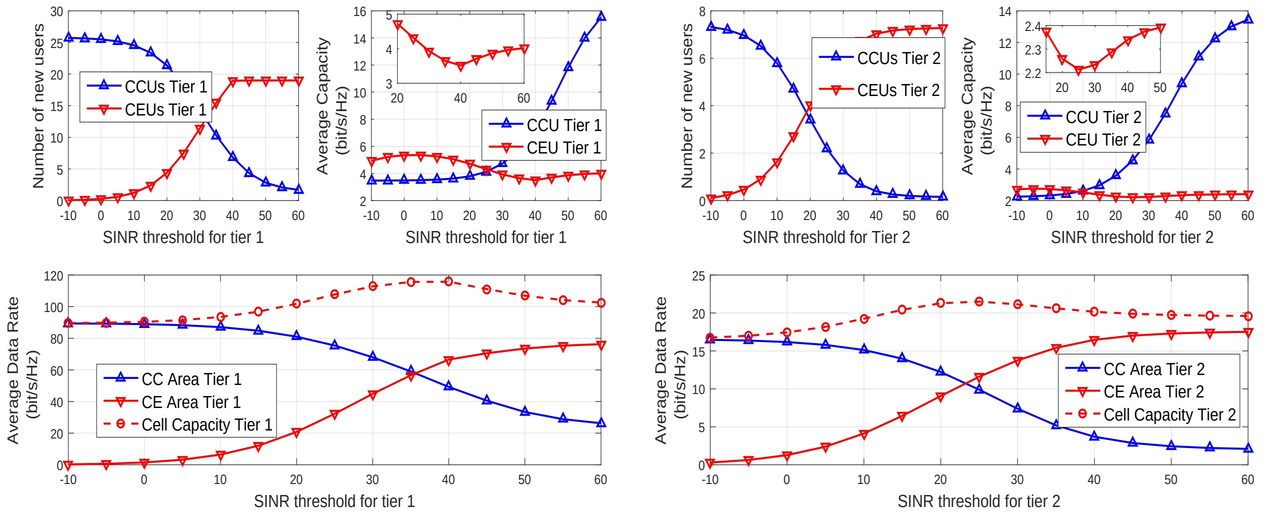
In this section, the average number of CCUs and CEUs as well as their average data rates are analysed for different values of SINR thresholds for Tier-1 and Tier-2. Since the changes in the SINR threshold for a given tier has a small impact on the performance of other tiers, either the performance of Tier-1 or Tier-2 is plotted in each of Figures 9 and 10.

It is observed from Figures 9 and 10 that there are opposite trends between the numbers of CCUs and CEUs in a given cell. When the SINR threshold increases, more users are served as CEUs and ther is a decline in the average number of CCUs. Since, at a given timeslot, each BS in Tier-k can only serve a maximum of $N_k^{(e)}$ CEUs in which $N_k^{(e)}$ is the average number of CE RBs in the corresponding cell, the average number of served users keeps



(a) Tier-1 (b) Tier-2

Fig. 9 *Strict FR*: Performance of Tier-1 and Tier-2 with different values of SINR thresholds



(a) Tier-1 (b) Tier-2

Fig. 10 *Soft FR*: Performance of Tier-1 and Tier-2 with different values of SINR thresholds

constant at 9 in Figure 9(a) and 19 in Figure 10(a) when the SINR threshold is greater than 35 dB and 40 dB, respectively.

From the CEU view, although each CEU experiences a higher level of the ICI when more users are served as the CEUs, the average data rate of the CEU fluctuates and can be divided into three regimes. This finding is very interesting and was not found in the previous works such as [4,7] since in those works, the average number of RBs and number of BSs were not considered in analysis approach. In *the first regime*, which corresponds to low values of the SINR threshold ($T_1 < 5$ dB and $T_2 < 0$ dB), the average data rate moderately increases to a peak value, e.g. of 4.534 (bit/s/Hz) at $T_1 = 5$ dB under the Strict FR. As can be seen from these two figures, a very small number of users are served as CEUs. Thus, the probability that two BSs use the same RB at the same time (called interfering probability), and consequently the ICI are infinitesimal. In the case of Strict FR with $T_2 = 0$ dB, the average number of CEUs is 0.1811 and thus the interfering probability is $\epsilon_k^{(e)} \epsilon_j^{(e)} = 1.46.10^{-4}$. Therefore, the effect of ICI in this case can be neglected and the average data rate of the CEU mainly depends on the SNR. Therefore, when more users with high SINRs (high data rates) are forced to be CEUs, the average data rate of the CEU increases.

In *the second regime*, which corresponds to medium values of the SINR threshold (5 $dB < T_1, T_2 < 35$ dB in the case of Strict FR; 0 $dB < T_1 < 40$ dB and 0 $dB < T_2 < 25$ dB in the case of Soft FR), the average number of CEUs increases rapidly as shown in Figure 9(a) and Figure 10(a) which leads to a significant increase in the interfering probability. As shown in Figure 9(a), when T_2 changes from 0 dB to 10 dB, the interfering probability increases 19.5 times from $1.46.10^{-4}$ to 0.0028. Hence, the ICI has considerably a negative impact on the user performance which results in a decline in the CEU's capacity.

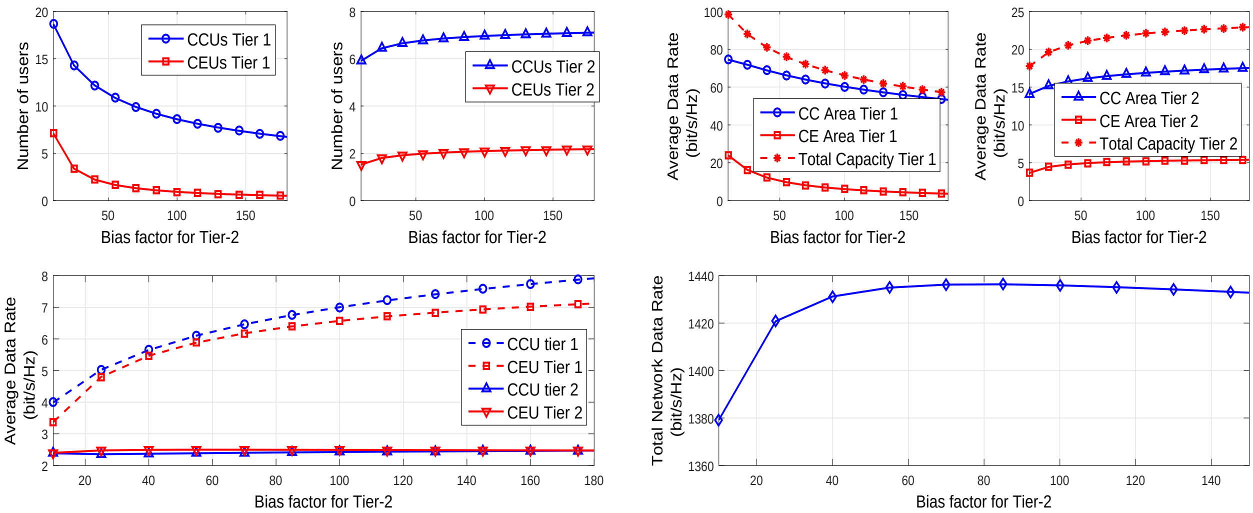
In *the third regime* corresponds to high values of the SINR threshold as highlighted on figures. For Tier-1 of both Strict FR and Soft FR, the average number of CEUs exceeds the average number of RBs, thus each BS has to transmit on all allocated CE RBs to serve the associated CEUs and this creates ICI to all CEUs in adjacent cells. This can be considered as the worst case of the CEU as it suffers from the ICI coming from all BSs transmitting at the CE power. For Tier-2, the interfering probability remains at high values when the SINR threshold changes. For example, under the Strict FR, when T_2 increases from 35 dB to 45 dB, the interfering probability increases by 1.33 times from 0.1669 to 0.2225 which is much smaller to 19.5 times when T_2 increases from 0 dB to 10 dB. Therefore, the ICI has small changes. Consequently, when more users with high SINR are pushed to be CEUs, the effect of the ICI on the CEU is unlikely to change but the average SINR of CEUs increases. Hence, the average data rate of the CEU is pushed up. It is pointed out that the average data rate of the CEU in Tier-2 in the third regime is always smaller than that in the first regime since more users are served as CEUs, hence more severe ICI effect on the CEU. Meanwhile in Tier-1, due to the constant ICI in the third regime, the average data rate of the CEU may be higher than that in the first regime.

From the CCU view, having more CEUs means that less users are defined as CCUs in the network. This can bring more opportunities for each CCU to be allocated a RB in order to avoid reusing frequency and thus the ICI can reduce, especially under the Strict FR where the CCU is only affected by the ICI from the BSs transmitting at the CC power. As indicated in Figure 9, the average data rate of the CCU in both Tier-1 and Tier-2 dramatically goes up from 2.972 (bit/s/Hz) to 12.29 (bit/s/Hz) and from 2.314 (bit/s/Hz) to 10.89 (bit/s/Hz) when the average number of CCUs falls from 24.54 to 1.22 and from 4.965 to 0.0268 at $T_2 = 10$ dB and $T_2 = 50$ dB, respectively. For the case of Soft FR, since the CCU suffers the ICI from BSs transmitting at the CC and CE powers, pushing more users to be CEUs can increase the ICI from BSs transmitting at the CE power. However, this negative impact is not significant and can counterbalance with a decrease in the ICI from BSs transmitting at the CC power since the density of BSs transmitting at the CC power in Tier-j is $(\Delta_1 - 1)$ times that of BSs transmitting at the CE power. Therefore, it is expected to observe from Figure 10 that the average data rate of the CCU under the Soft FR also dramatically goes up but a little bit slower than that under the Strict FR.

Although the average data rate of the CCU increases when the SINR threshold increases, the rapid decline in the average number of CCUs leads to an decrease of the average data rate of the CC Area in each cell of each Tier for both Strict FR and Soft FR. Meanwhile, the average data rate of CE Area steadily increases. Therefore, the average data rate per cell reaches to the peak before passing a significant decline. Thus, the optimal value of the SINR threshold for Tier-1 and Tier-2 can be found when the average data rate is at the peak. For example, under the Soft FR $T_1 = 40$ dB and $T_2 = 25$ dB are chosen in order to achieve the maximum average data rate per cell of 115.8 (bit/s/Hz) for Tier-1 and 21.5 (bit/s/Hz) for Tier-2.

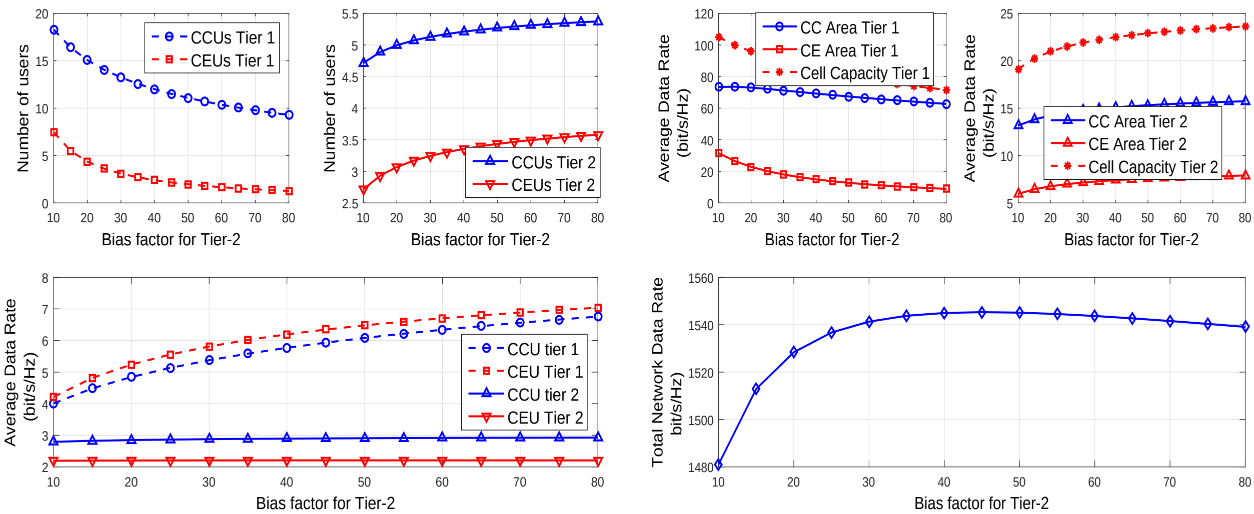
6.4 Bias Association

The bias factor is used for handover users from Tier-1 to Tier-2 to maintain load balancing between tiers in the network.



(a) Average number of new users per cell and average capacity (b) Average Capacity of CC, CE Area and overall network

Fig. 11 *Strict FR*: Average number of users and data rate with different values of the bias factor for Tier-2, B_2



(a) Average number of new users per cell and average capacity (b) Average data rate of CC, CE Area and overall network

Fig. 12 *Soft FR*: Average number of users and data rate with different values of the bias factor for Tier-2, B_2

It is observed from Figures 11 and 12 that when the bias factor B_2 for Tier-2 increases, more users from Tier-1 are pushed to Tier-2. However, since the density of BSs in Tier-2 is 10 times that of Tier-1, the changes in the average number of users in Tier-1 only result in a small change in the average number of users in Tier-1. Take the case of Soft FR as an example, the average number of users in Tier-1 goes down by around 50% from 25.77 at $B_2 = 10$ to 13.045 at $B_2 = 50$, while the average number of users in Tier-2 only goes up by 14.6% from 7.43 to 8.70. As discussed in Section 6.3, these changes reflect a rapid downward trend in the ICI from BSs in Tier-1 and a slow upward trend in the ICI from BSs in Tier-2. In other words, the ICI in this case is mitigated. Therefore, as illustrated in Figure 11(a) and Figure 12(a), the average data rates of both CCU and CEU in Tier-1 increase moderately while the changes in Tier-2 are marginal.

The total network data rate is obtained by finding the sum of cell data rates of all cells in the network. It is observed from Figure 11(b) and Figure 12(b) that the total network data rate for both Strict FR and Soft FR reach the peak at 1437 (bit/s/Hz) and 1457 (bit/s/Hz) before passing a steady decline. Hence, the optimal values of bias factor can be selected at 70 for Strict FR and 40 for Soft FR.

7 Conclusions

In this paper, we derived the average coverage probabilities and average data rates of CCUs and CEUs as well as CC and CE Areas in the heterogeneous network using Strict FR and Soft FR. We proposed an analytical approach which is more accurate than the previous works. Through the analytical results, the effects of the SINR threshold and the bias factor on the network performance were clearly presented. While the CCU average data rate increases when the SINR threshold increases, the CEU average data rate fluctuates and can be partitioned into 3 regimes corresponding to 3 ranges of the SINR threshold values. Moreover, an increase in the bias factor can improve the average data rates of both CEU and CCU in all tiers for both Strict FR and Soft FR. Thus, we presented an approach to find an optimal value of the SINR threshold and the bias factor to obtain the maximum network capacity. Furthermore, the analytical results indicated that the proposed model can reduce the BS power consumption while improving the network data rates.

A Theorem 2 - CCU classification

The average probability where the user is served as a CCU in Tier-k of Soft FR is given by

$$\begin{aligned}\mathbb{P}^{(nc)}_{Soft,k}(T_k) &= \mathbb{P}\left(\frac{P_k g'_k r_k^{\alpha_k}}{\sigma_G^2 + I} > T_k\right)\\ &\overset{(a)}{=} e^{-\frac{T_k r_k^{\alpha_k}}{SNR_k}}\,\mathbb{E}\left[e^{-\frac{T_k r_k^{\alpha_k} I^{(u)}_{Sof}}{P_k}}\right]\end{aligned} \tag{28}$$

where $SNR_k = \frac{P_k}{\sigma^2}$; (a) follows the assumption that the channel fading has Rayleigh distribution.

By substituting Equation 8, the expectation can be presented as the product of $\mathbb{E}(\phi|r_k)$ and $\mathbb{E}(1|r_k)$ where

$$\mathbb{E}(\phi_j|r_k) = \prod_{j=1}^{K} \mathbb{E}\left\{ \prod_{z_e \in \theta_j^{(e)}} \left[1 - E[\tau_k^{(e)} \tau_j^{(z_e)}] \left(1 - e^{-s_{jz_e} g_{jz_e} r_{jz_e}^{-\alpha_j}} \right) \right] \right\} \tag{29a}$$

$$\mathbb{E}(1|r_k) = \prod_{j=1}^{K} \mathbb{E}\left\{ \prod_{z_c \in \theta_j^{(c)}} \left[1 - E[\tau_k^{(c)} \tau_j^{(z_c)}] \left(1 - e^{-s_{jz_c} g_{jz_c} r_{jz_c}^{-\alpha_j}} \right) \right] \right\} \tag{29b}$$

in which $T_k \frac{\phi_j P_j}{P_k} r_k^{\alpha_k} = s_{jze}$ and $T_k \frac{P_j}{P_k} r_k^{\alpha_k} = s_{jzc}$.

Evaluating $\mathbb{E}(\phi|r_k)$ and since g_{jz_e} is exponential RV, we have

$$\begin{aligned}\mathbb{E}(\phi_j|r_k) &= \prod_{j=1}^{K} \mathbb{E}\left\{ \prod_{z_e \in \theta_j^{(e)}} \left[1 - \epsilon_k^{(oe)} \epsilon_j^{(oe)} \left(1 - \frac{1}{1 + s_{jz_e} r_{jz_e}^{-\alpha_j}} \right) \right] \right\} \\ &\overset{(b)}{=} \prod_{j=1}^{K} e^{-2\pi \epsilon_k^{(oc)} \epsilon_j^{(oe)} \lambda_j^{(e)} \int_{\sqrt{C_j} r^{\alpha_k/\alpha_j}}^{\infty} \left[1 - \frac{1}{1+s_{jz_e} r_{jz_e}^{-\alpha_j}} \right] r_{jze} dr_{jze}} \\ &\overset{(c)}{=} e^{-\pi \epsilon_k^{(oc)} \sum_{j=1}^{K} \lambda_j^{(e)} C_j r_k^{2\alpha_k/\alpha_j} \int_0^1 \frac{\epsilon_j^{(oe)}}{\frac{1}{T_k} \frac{B_j}{\phi_j B_k} x^{2-\alpha_j/2} + x^2} dx}\end{aligned} \tag{30}$$

in which (a) is obtained by using the properties of PPP probability generating function and $r_j > \sqrt{C_j} r^{\alpha_k/\alpha_j}$; (c) is obtained by employing a change of variables $x = C_j r_k^{\frac{2\alpha_k}{\alpha_j}} r_{jz_e}^{-2}$.

$$\text{Similarity, } \mathbb{E}(1|r_k) = e^{-\pi \epsilon_k^{(oc)} \sum_{j=1}^{K} \lambda_j^{(c)} C_j r_k^{2\alpha_k/\alpha_j} \int_0^1 \frac{\epsilon_j^{(oc)}}{\frac{1}{T_k} \frac{B_j}{B_k} x^{2-\alpha_j/2} + x^2} dx} \tag{31}$$

Substituting $\mathbb{E}(\phi_k|r_k)$ and $\mathbb{E}(1|r_k)$ into Equation 28, then the results in Theorem 4.2 is obtained

B Theorem 3 - CCU under Strict FR

The average coverage probability of a CCU under the Strict FR is given by

$$P_{Str,k}^{(e)}(T_k, \hat{T}_k) \overset{(a)}{=} \frac{\mathbb{P}\left(SINR(1, r_k) > \hat{T}_k, SINR^{(o)}(1, r_k) > T_k \right)}{\mathbb{P}\left(SINR^{(o)}(1, r_k) > T_k \right)} \tag{32}$$

$$\overset{(b)}{=} \frac{\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2} e^{-\frac{r_k^{\alpha_k}}{SNR_k}(\hat{T}_k + T_k)} \mathbb{E}\left[e^{-\left(\frac{T_k I_{Str}^{(oc)}}{P_k} + \frac{\hat{T}_k I_{Str}^{(c)}}{P_k} \right) r_k^{\alpha_k}} \right] dr_k}{\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2} \mathbb{E}\left[e^{-\frac{T_k}{P_k}\left(I_{Str}^{(oc)} + \sigma^2 \right) r_k^{\alpha_k}} \right] dr_k} \tag{33}$$

in which (a) follows the Bayes rules and (b) follows the assumption that the fading channel coefficients are independent RV and conditioning on r_k.

The expectation in the numerator can be computed based on the approach in [5]

$$\mathbb{E}\left[e^{-\left(\frac{T_k I_{Str}^{(oc)}}{P_k}+\frac{\hat{T}_k I_{Str}^{(c)}}{\phi_k P_k}\right) r_k^{\alpha_k}}\right] = \mathbb{E}\left[\prod_{z_c\in\theta_j^{(c)}} e^{-\tau_j^{(oc)}\tau_j^{(oz_c)} g_{jz_c}^{(o)} s_{z_c}} e^{-\tau_j^{(c)}\tau_j^{(z_c)} g_{jz_c}\hat{s}_{z_c}}\right]$$
$$= \mathbb{E}\left[\prod_{z_c\in\theta_j^{(c)}} \left(1-\epsilon_k^{(oc)}\epsilon_j^{(oc)}\frac{s_{z_c}}{1+s_{z_c}}\right)\left(1-\epsilon_k^{(c)}\epsilon_j^{(c)}\frac{\hat{s}_{z_e}}{1+\hat{s}_{z_e}}\right)\right] \tag{34}$$

where $T_k\frac{P_j}{P_k}\frac{r_{jz_c}^{-\alpha_j}}{r_k^{-\alpha_k}} = s_{z_c}$ and $\hat{T}_k\frac{P_j}{P_k}\frac{r_{jz_c}^{-\alpha_j}}{r_k^{-\alpha_k}} = \hat{s}_{z_c}$; and the second equality follows the properties of the indicator function.

Using the properties of the PPP probability generating function with $r_{jz_c} > \sqrt{C_j} r^{\alpha_k/\alpha_j}$ and employing a change of variable $x = C_j r_k^{\frac{2\alpha_k}{\alpha_j}} r_{jz_c}^{-2}$;, then

$$\mathbb{E}\left[e^{-\left(\frac{T_k I_{Str}^{(oc)}}{P_k}+\frac{\hat{T}_k I_{Str}^{(c)}}{\phi_k P_k}\right) r_k^{\alpha_k}}\right] = e^{-2\pi\lambda_j C_j r_k^{2\alpha_k/\alpha_j}\left(\epsilon_k^{(oc)}\upsilon_j^{(c)}(T_k)+\epsilon_k^{(c)}\upsilon_j^{(c)}(\hat{T}_k)-\epsilon_k^{(oc)}\epsilon_k^{(c)}\kappa(T_k,\hat{T}_k)dx\right)} \tag{35}$$

in which $\upsilon_j^{(oc)}(T_k) = \int_0^1 \frac{\epsilon_j^{(oc)}}{\frac{1}{T_k}\frac{B_j}{B_k}x^{2-\alpha_j/2}+x^2}dx$; $\upsilon_j^{(c)}(\hat{T}_k) = \int_0^1 \frac{\epsilon_j^{(c)}}{\frac{1}{\hat{T}_k}\frac{B_j}{B_k}x^{2-\alpha_j/2}+x^2}dx$ and $\kappa^{(c)}(T_k,\hat{T}_k) = \int_0^1 \frac{\epsilon_j^{(oc)}\epsilon_j^{(c)}}{x^2\left(\frac{1}{T_k}\frac{B_j}{B_k}x^{-\alpha_j/2}+1\right)\left(\frac{1}{\hat{T}_k}\frac{B_j}{B_k}x^{-\alpha_j/2}+1\right)}$

C Theorem 4 - CEU under Strict FR

. The average coverage probability of a CEU under the Strict FR is evaluated by

$$P_{Str,k}^{(e)}(\hat{T}_k,T_k) = \frac{\mathbb{P}\left(SINR(\phi_k,r_k)>\hat{T}_k, SINR^{(o)}(1,r_k)<T_k\right)}{\mathbb{P}\left(SINR^{(o)}(1,r_k)<T_k\right)}$$
$$= \frac{\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2}\mathbb{E}\left[e^{-\frac{\hat{T}_k r_k^{\alpha_k}}{\phi_k P}\left(I_{Str}^{(e)}+\sigma^2\right)}\left(1-e^{-\frac{T_k}{P_k}\left(I_{Str}^{(oc)}+\sigma^2\right)r_k^{\alpha_k}}\right)\right]dr_k}{1-\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2}\mathbb{E}\left[e^{-\frac{T_k}{P_k}\left(I_{Str}^{(oc)}+\sigma^2\right)r_k^{\alpha_k}}\right]dr_k} \tag{36}$$

The expected value of the numerator can be separated into two expectations in which the first one is evaluated using the same approach as in Appendix A, i.e, $\mathbb{E}\left[e^{-\frac{\hat{T}_k}{\phi_k P_k}\left(I_{Str}^{(e)}+\sigma^2\right)r_k^{\alpha_k}}\right] = e^{-\frac{\hat{T}_k r_k^{\alpha_k}}{\phi_k SNR_k}-\pi\lambda_k\epsilon_k^{(e)2}r_k^2\upsilon_j^{(e)}(\hat{T}_k)}$, and the second one can be computed in the following steps

$$= \mathbb{E}\left[\prod_{z_c\in\theta_j^{(c)}} e^{-\tau_j^{(oc)}\tau_j^{(oz_c)} g_{jz_c}^{(o)} s_{z_c}} \prod_{z_e\in\theta_j^{(e)}} e^{-\tau_j^{(e)}\tau_j^{(z_e)} g_{jz_e}\hat{s}_{z_e}}\right]$$
$$= \mathbb{E}\left[\prod_{z_c\in\theta_j^{(c)}}\left(1-\epsilon_k^{(oc)}\epsilon_j^{(oc)}\frac{s_{z_c}}{1+s_{z_c}}\right)\prod_{z_e\in\theta_j^{(e)}}\left(1-\epsilon_k^{(e)}\epsilon_j^{(e)}\frac{\hat{s}_{z_e}}{1+\hat{s}_{z_e}}\right)\right]$$

in which $T_k\frac{r_{jz_c}^{-\alpha_j}}{r_k^{-\alpha_k}} = s_{z_c}$ and $\hat{T}_k\frac{\phi_j}{\phi_k}\frac{r_{jz_e}^{-\alpha_j}}{r_k^{-\alpha_k}} = \hat{s}_{z_e}$.

During the establishment phase, the user may experience the ICI from all BSs while the CEU is only affected by $\frac{1}{\Delta_j}$ BSs, thus $\theta_j^{(e)}$ is the subset of $\theta_j^{(c)}$. Furthermore, since in PPP, each point is stochastically independent to all other points, $\theta_j^{(e)}$ and $\theta_j^{(c)} \backslash \theta_j^{(e)}$ are independent in which the density of BSs of $\theta_j^{(c)} \backslash \theta_j^{(e)}$ is $\frac{\Delta_j - 1}{\Delta_j}\lambda_j$ and of $\theta_j^{(e)}$ is $\frac{1}{\Delta_j}\lambda_j$. Therefore, the expectation equals

$$
\begin{aligned}
=&\mathbb{E}_{\theta_j^{(c)} \backslash \theta_j^{(e)}}\left[\prod_{z_c \in \theta_j^{(c)} \backslash \theta_j^{(e)}} \left(1 - \epsilon_k^{(oc)} \epsilon_j^{(oc)} \frac{s_{z_c}}{1 + s_{z_c}}\right)\right] \\
&\mathbb{E}_{\theta_j^{(e)}}\left[\prod_{z_e \in \theta_j^{(e)}} \left(1 - \epsilon_k^{(oc)} \epsilon_j^{(oc)} \frac{s_{z_e}}{1 + s_{z_e}}\right)\left(1 - \epsilon_k^{(e)} \epsilon_j^{(e)} \frac{\hat{s}_{z_e}}{1 + \hat{s}_{z_e}}\right)\right]
\end{aligned}
\tag{37}
$$

in which $s_{z_e} = T_k \frac{r_{jz_e}^{-\alpha_j}}{r_k^{-\alpha_k}}$.

Employing the properties of the PPP probability generating function with $r_{jz_c} > \sqrt{C_j} r^{\alpha_k/\alpha_j}$ and the changes of variables $x = C_j r_k^{\frac{2\alpha_k}{\alpha_j}} r_{jz_c}^{-2}$ for the first part and $x = C_j r_k^{\frac{2\alpha_k}{\alpha_j}} r_{jz_e}^{-2}$ for the second part, the expectation equals

$$
= e^{-2\pi\lambda_j C_j r_k^{2\alpha_k/\alpha_j}\left[\epsilon_k^{(oc)} \upsilon_j^{(oc)}(T_k) + \frac{1}{\Delta_j}\epsilon_k^{(e)} \upsilon_j^{(e)}(\hat{T}_k) - \frac{1}{\Delta_j}\epsilon_k^{(oc)}\epsilon_k^{(e)}\kappa(T_k, \hat{T}_k)\right] dx}
\tag{38}
$$

in which $\upsilon_j^{(e)}(\hat{T}_k) = \int_0^1 \frac{\epsilon_j^{(e)}}{\frac{1}{\hat{T}_k}\frac{\phi_k}{\phi_j}\frac{B_j}{B_k} x^{2-\alpha_j/2} + x^2} dx$

and $\kappa(T_k, \hat{T}_k) = \int_0^1 \frac{\epsilon_j^{(oc)}\epsilon_j^{(c)}}{x^2\left(\frac{1}{T_k}\frac{B_j}{B_k} x^{-\alpha_j/2} + 1\right)\left(\frac{1}{\hat{T}_k}\frac{\phi_k}{\phi_j}\frac{B_j}{B_k} x^{-\alpha_j/2} + 1\right)}$; and $\upsilon_j^{(oc)}(T_k)$ is defined in Equation (35).

The expectation of the denominator in Equation 33 is computed by Appendix A, then

$$
\mathbb{E}\left[e^{-\frac{T_k}{\phi_k P_k}\left(I_{Str}^{(oe)} + \sigma^2\right) r_k^{\alpha_k}}\right] = e^{-\frac{T_k r_k^{\alpha_k}}{SNR_k}} e^{-\pi \sum_{j=1}^{K} \lambda_j C_j \epsilon_k^{(oc)} \upsilon_j^{(oc)}(T_k) r_k^{\frac{2\alpha_k}{\alpha_j}}}
\tag{39}
$$

Substituting Equations (38) and (39) into Equation 33, Theorem 4 is proved.

D Theorem 5 - CCU under Soft FR

The coverage probability of a CCU under the Soft FR network is obtained by

$$
\begin{aligned}
P_c^{(c)}(T_k, \hat{T}_k) &= \frac{\mathbb{P}\left(\frac{P_k g_k' r_k^{-\alpha_k}}{\sigma_G^2 + I_{Soft}^{(c)}} > \hat{T}_k, \frac{P_k g_k r_k^{-\alpha_k}}{\sigma_G^2 + I_{Soft}^{(oc)}} > T_k\right)}{\mathbb{P}\left(\frac{P_k g_k r_k^{-\alpha_k}}{\sigma_G^2 + I_{Soft}^{(oc)}} > T_k\right)} \\
&= \frac{\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2} e^{-\frac{r_k^{\alpha_k}}{SNR_k}\left(\hat{T}_k + T_k\right)} \mathbb{E}\left[e^{-\frac{r_k^{\alpha_k}}{P_k}\left(\hat{T}_k I_{Soft}^{(c)} + T_k I_{Soft}^{(oc)}\right)}\right] dr_k}{\int_0^\infty 2\pi\lambda_k r_k e^{-\pi\lambda_k r_k^2} \mathbb{E}\left[e^{-\frac{T_k}{P_k}\left(\sigma_G^2 + I_{Soft}^{(oc)}\right)}\right] dr_k}
\end{aligned}
\tag{40}
$$

Substituting Equation (8) into Equation (40), the second element of the integrand in the numerator of (40) can be evaluated by using the approach in [5]:

$$\stackrel{(c)}{=} \prod_{j=1}^{K} \mathbb{E}\left[\prod_{z_c\in\theta_j^{(c)}} e^{-\left(\hat{T}_k\tau_k^{(c)}\tau_j^{(z_c)}g'_{jz_c}+T_k\tau_k^{(oc)}\tau_j^{(oz_c)}g_{jz_c}\right)\frac{P_j}{P_k}\frac{r_{jz_c}^{-\alpha_j}}{r_k^{-\alpha_k}}}\right] \mathbb{E}\left[\prod_{z_e\in\theta_j^{(e)}} e^{-\left(\hat{T}_k\tau_k^{(e)}\tau_j^{(z_e)}g'_{jz_e}+T_k\tau_k^{(oe)}\tau_j^{(oz_e)}g_{jz_e}\right)\frac{\phi_j P_j}{P_k}\frac{r_{jz_e}^{-\alpha_j}}{r_k^{-\alpha_k}}}\right] \tag{41}$$

in which (c) follows the fact that in a PPP network model, each BS is distributed stochastically independent to all the BSs and the fading power gains are independent RVs.

The first element of Equation (41) can be evaluated as follows

$$\begin{aligned}
&\stackrel{(d)}{=} \mathbb{E}\left[\prod_{z_c\in\theta_j^{(c)}}\left(1-\mathbb{E}\left[\tau_k^{(oc)}\tau_j^{(oz_c)}\right]\left(1-e^{-s_{jz_c}g_{jz_c}r_{jz_c}^{-\alpha_j}}\right)\right)\left(1-\mathbb{E}\left[\tau_k^{(c)}\tau_j^{(z_c)}\right]\left(1-e^{-\hat{s}_{jz_c}g_{jz_c}r_{jz_c}^{-\alpha_j}}\right)\right)\right]\\
&\stackrel{(e)}{=} \mathbb{E}\left[\prod_{z_c\in\theta_j^{(c)}}\left(1-\epsilon_k^{(oc)}\epsilon_j^{(oc)}\left(1-\frac{1}{1+s_{jz_c}r_{jz_c}^{-\alpha_j}}\right)\right)\left(1-\epsilon_k^{(c)}\epsilon_j^{(c)}\left(1-\frac{1}{1+\hat{s}_{jz_c}r_{jz_c}^{-\alpha_j}}\right)\right)\right]\\
&\stackrel{(f)}{=} e^{-\frac{2\pi\lambda_j C_j(\Delta_j-1)}{\Delta_j}r_k^{\frac{2\alpha_k}{\alpha_j}}\int_0^1\left[\frac{\epsilon_k^{(oc)}\epsilon_j^{(oc)}}{\frac{1}{T_k}\frac{B_j}{B_k}x^{2-\alpha_j/2}+x^2}+\frac{\epsilon_k^{(c)}\epsilon_j^{(c)}}{\frac{1}{\hat{T}_k}\frac{B_j}{B_k}x^{2-\alpha_j/2}+x^2}-\frac{\epsilon_k^{(oc)}\epsilon_k^{(c)}\epsilon_j^{(oc)}\epsilon_j^{(c)}}{x^2\left(1+\frac{1}{T}\frac{B_j}{B_k}x^{-\alpha_j/2}\right)\left(1+\frac{1}{\hat{T}}\frac{B_j}{B_k}x^{-\alpha_j/2}\right)}\right]dx}
\end{aligned} \tag{42}$$

where (d) follows the properties of the indicator function and by letting $T\frac{P_j}{P_k}r_k^{\alpha_k} = s_{jze}$ and $\hat{T}_k\frac{P_j}{P_k}r_k^{\alpha_k} = \hat{s}_{jze}$; (e) follows the assumption that the channel fading has a Rayleigh distribution; (f) follows the properties of the PPP probability generating function with $r_{jz_c} > \sqrt{C_j}r^{\alpha_k/\alpha_j}$ and by letting $x = C_j r^{\frac{2\alpha_k}{\alpha_j}} r_{jz_e}^{-2}$.

Similarly, the second expectation of Equation (41) is obtained by

$$e^{-\frac{2\pi\lambda_j C_j}{\Delta_j}r_k^{\frac{2\alpha_k}{\alpha_j}}\int_0^1\left[\frac{\epsilon_k^{(oc)}\epsilon_j^{(oe)}}{\frac{1}{T_k}\frac{B_j}{\phi_j B_k}x^{2-\alpha_j/2}+x^2}+\frac{\epsilon_k^{(c)}\epsilon_j^{(e)}}{\frac{1}{\hat{T}_k}\frac{B_j}{\phi_j B_k}x^{2-\alpha_j/2}+x^2}-\frac{\epsilon_k^{(oc)}\epsilon_k^{(c)}\epsilon_j^{(oe)}\epsilon_j^{(e)}}{x^2\left(1+\frac{1}{T}\frac{B_j}{\phi_j B_k}x^{-\alpha_j/2}\right)\left(1+\frac{1}{\hat{T}}\frac{B_j}{\phi_j B_k}x^{-\alpha_j/2}\right)}\right]dx} \tag{43}$$

Substituting Equations (42) and (43) into Equation (40), and notice that the denominator of Equation 40 is evaluated in Appendix A, Theorem 5 is proved.

References

References

1. 3GPP-TSG-RAN-R1-050738, Interference mitigation Considerations and Results on Frequency Reuse (September, 2005, Siemens).
2. F. Mhiri, K. Sethom, R. Bouallegue, A survey on interference management techniques in femtocell self-organizing networks, Journal of Network and Computer Applications 36 (1) (2013) 58 – 65. `doi:https://doi.org/10.1016/j.jnca.2012.04.021`. URL `http://www.sciencedirect.com/science/article/pii/S1084804512001142`

3. S. C. Lam, K. S. Sandrasegaran, Performance analysis of fractional frequency reuse in uplink random cellular networks, Physical Communication (sept 2017). doi:https://doi.org/10.1016/j.phycom.2017.09.008.
4. T. D. Novlan, Fractional frequency reuse for multi-tier cellular networks, 2012.
URL https://repositories.lib.utexas.edu/handle/2152/ETD-UT-2012-05-5184
5. T. D. Novlan, R. K. Ganti, A. Ghosh, J. G. Andrews, Analytical Evaluation of Fractional Frequency Reuse for OFDMA Cellular Networks, IEEE Trans. Wireless Commun. 10 (2011) 4294–4305.
6. H. Zhuang, T. Ohtsuki, A model based on Poisson point process for downlink K tiers fractional frequency reuse heterogeneous networks, Physical Communication Volume 13, Part B (Special Issue on Heterogeneous and Small Cell Networks) (2014) 3–12.
7. S. Kumar, S. Kalyani, K. Giridhar, Optimal design parameters for coverage probability in fractional frequency reuse and soft frequency reuse, IET Communications 9 (10) (2015) 1324–1331.
8. H.-S. Jo, Y. J. Sang, P. Xia, J. Andrews, Heterogeneous Cellular Networks with Flexible Cell Association: A Comprehensive Downlink SINR Analysis, IEEE Trans. Wireless Commun. 11 (10) (2012) 3484–3495. doi:10.1109/TWC.2012.081612.111361.
9. J. Moller, R. P. Waagepeterse, Statistical Inference and Simulation for Spatial Point Processes, CRC Press, 2003.
10. F. Baccelli, M. Klein, M. Lebourges, S. Zuyev, Stochastic geometry and architecture of communication networks, Telecommunication Systems 7 (1) (1997) 209–227. doi:10.1023/A:1019172312328.
URL http://dx.doi.org/10.1023/A:1019172312328
11. T. D. Novlan, R. K. Ganti, A. Ghosh, J. G. Andrews, Analytical Evaluation of Fractional Frequency Reuse for Heterogeneous Cellular Networks, IEEE Trans. Commun. 60 (7) (2012) 2029–2039. doi:10.1109/TCOMM.2012.061112.110477.
12. S. C. Lam, K. Sandrasegaran, P. Ghosal, Performance analysis of frequency reuse for ppp networks in composite rayleigh–lognormal fading channel, Wireless Personal Communications (Apr 2017). doi:10.1007/s11277-017-4215-2.
13. Y. Lin, W. Bao, W. Yu, B. Liang, Optimizing User Association and Spectrum Allocation in HetNets: A Utility Perspective, IEEE J. Sel. Areas Commun. 33 (6) (2015) 1025–1039. doi:10.1109/JSAC.2015.2417011.
14. Y. Lin, W. Yu, Optimizing user association and frequency reuse for heterogeneous network under stochastic model, in: 2013 IEEE Global Commun. Conf.(GLOBECOM), 2013, pp. 2045–2050. doi:10.1109/GLOCOM.2013.6831376.
15. D. Cao, S. Zhou, Z. Niu, Improving the Energy Efficiency of Two-Tier Heterogeneous Cellular Networks through Partial Spectrum Reuse, IEEE Trans. Wireless Commun. 12 (8) (2013) 4129–4141. doi:10.1109/TWC.2013.052213.121751.
16. S. E. Sagkriotis, A. D. Panagopoulos, Optimal FFR Policies: Maximization of Traffic Capacity and Minimization of Base Station's Power Consumption, IEEE Wireless Commun. Lett. 5 (1) (2016) 40–43. doi:10.1109/LWC.2015.2491943.
17. L. Su, C. Yang, C. L. I, Energy and Spectral Efficient Frequency Reuse of Ultra Dense networks, IEEE Trans. Wireless Commun. 15 (8) (2016) 5384–5398. doi:10.1109/TWC.2016.2557790.
18. Z. Ning, Q. Song, L. Guo, X. Kong, A novel adaptive spectrum allocation scheme for multi-channel multi-radio wireless mesh networks, Journal of Network and Computer Applications 56 (Supplement C) (2015) 19 – 27. doi:https://doi.org/10.1016/j.jnca.2015.06.004.
URL http://www.sciencedirect.com/science/article/pii/S1084804515001289
19. 3GPP TS 36.213 version 8.8.0 Release 8, LTE; Evolved Universal Terrestrial Radio Access (E-UTRA); Physical layer procedures, 2009.
20. HUAWEI TECHNOLOGIES CO., LTD, eLTE2.2 DBS3900 LTE FDD Basic Feature Description, 2014.